DEPARTMENT OF THE NAVY
DAVID TAYLOR MODEL BASIN

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STRUCTURAL ANALYSIS AND DESIGN CONSIDERATIONS FOR CYLINDRICAL PRESSURE HULLS

by

John G. Pulos

STRUCTURAL MECHANICS LABORATORY

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FOREWORD

This report represents a summary of structural research efforts by many groups within and outside the United States Navy. Significant contributions during the past 15 years have been made by personnel at the David Taylor Model Basin, under the direction of Dr. E. Wenk, Jr., Mr. E. E. Johnson, Commander S. R. Heller, U.S.N., and Dr. A. H. Keil, and by submarine hull designers at the Bureau of Ships, notably, Messrs. J. Vasta, O. Oakley, F. Dunham, and A. Stavovy. The results of theoretical studies on shell structures conducted at the Polytechnic Institute of Brooklyn under financial support by the Office of Naval Research and the Bureau of Ships have made available to the naval architect some well-known methods of analysis which have proven very successful in the field of thin-walled aircraft structures; these investigations were under the general supervision of Professors N. J. Hoff and V. L. Salerno and are presently being continued under the supervision of Professor J. Kempner. Finally, our British counterparts at the Naval Construction Research Establishment, Dunfermline, Scotland, namely, Mr. S. R. Kendrick and Dr. A. Bryant formerly at NCRE have also made significant contributions toward a better understanding of the behavior and more scientific design of submarine hull structures. Throughout the discussion, recognition is given to those individuals who have made specific contributions in this field.

This paper is an extension of a series of lectures presented by the

author as part of a short course entitled, "Analysis and Design of Modern Pressure Vessels," held at the University of California in Los Angeles during 16 - 27 July, 1962.

NOTATION*

R	Radius to median surface of the cylindrical shell
h	Uniform thickness of the shell
L_f	Center-to-center spacing of reinforcing ring frames
b	Faying width of ring frames in contact with shell
$L = L_f - b$	Unsupported shell length between adjacent ring frames
A_f	Actual cross-sectional area of ring frames
p	Applied hydrostatic pressure
E	Modulus of elasticity
ν	Poisson ratio
$D = \dfrac{Eh^3}{12(1-\nu^2)}$	Flexural rigidity of shell
$w(x)$	Axisymmetric elastic radial deflection
x	Axial coordinate along length of cylinder
A, B, C, F	Arbitrary constants
$\theta = \sqrt[4]{3(1-\nu^2)}\,\dfrac{L}{\sqrt{Rh}}$	Shell flexibility parameter
$\gamma \equiv p/p^*$	Measure of beam-column effect
R_{cg}	Radius to centroid of ring frame
σ	Normal stress
$\alpha = \dfrac{A_f}{L_f h}$	Ratio of ring-frame area to cross-sectional area of one bay of shell
$\beta = \dfrac{b}{L_f}$	Ratio of ring-frame faying width to frame spacing
F_1, F_2, F_3, F_4, F_5	Functions of pressure loading and geometry of shell

* Basic symbols are given in the order in which they appear in the text and figures.

σ_y	Yield strength of shell material
m	Number of half waves in longitudinal direction of buckled cylinder
$u(x,\theta)$;$v(x,\theta)$;$w(x,\theta)$	Additional displacements in axial, circumferential, and radial directions, respectively, for asymmetric buckling
$S = R\theta$	Circumferential coordinate along periphery of cylinder
n	Number of full waves in circumferential direction of buckled cylinder
L_b	Overall length of stiffened cylindrical shell
I_f	Moment of inertia of ring frame cross section
I_{xo}	Moment of inertia about longitudinal axis through centroid of ring frame
I_{zo}	Moment of inertia about radial axis through centroid of ring frame

ABSTRACT

This report reviews the "state of the art" in the field of pressure hull structural analysis. Equations and formulas developed from considerations of thin-shell theory to describe the elastic and inelastic behavior of ring-stiffened cylindrical shells under the action of hydrostatic pressure are summarized.

No direct comparison between theory and experiment is included herein; however, only those analyses are included which represent the most up-to-date knowledge, in the opinion of David Taylor Model Basin structural analysts, and which have found firm experimental confirmation. This presentation should not be interpreted as representing an exhaustive review of all available stress and stability analyses for ring-stiffened cylindrical shells under hydrostatic pressure, but merely one which includes only those formulations which are essential and which have found extensive use in the design of cylindrical pressure hulls.

A few introductory remarks on some of the more promising high-strength hull materials and on new and untried hull configurations and construction techniques presently being investigated at the Model Basin are also presented.

INTRODUCTION

An interesting discussion of the role played by the submarine during World War II and the changes which have taken place in design philosophy since 1941 may be found in Reference 1* together with some comments on the circumstances which have led to these changes. A digest of fundamental principles underlying the design of submarine pressure hulls is given in Reference 2, from the point of view of the naval architect.

Wenk[3] has presented an outstanding report in the open literature on the feasibility of pressure hulls for ultradeep running submarines; he discusses some of the motivations for operating deeper, and presents the results of strength-weight calculations for different hull configurations and promising new hull materials. Although very little test data had been available with which to check the underlying assumptions and compare the various design concepts proposed by Wenk, his paper is an outstanding contribution to the literature in that it documents new ideas and construction techniques which may constitute the basis for realizing the deep-diving submersible. Other feasibility studies, notably those conducted about the same time at the Bureau of Ships and the David Taylor Model Basin, have led to similar results and to the formulation of detailed programs of research into new and untried hull structures and materials which are presently underway at the Model Basin. In Reference 4 Wenk sets forth some basic principles of pressure-hull engineering which, in

* References are given on page 134.

general, represent the state of knowledge as it was understood at the time his paper appeared.

The purpose of this report is to present and discuss in detail some of the physical concepts and mathematical formulations for the stress and buckling of cylindrical hull structures. No attempt will be made to discuss so-called design criteria or even to outline a design procedure because this could possibly infringe on the realm of classified information. Furthermore, it is beyond the scope of this short presentation to give extensive comparisons between experiment and theory, but it will be implied that only those theories and/or empirical formulas will be considered which are finding wide use and which represent the best knowledge of structural analysts at the Model Basin. It is assumed that the readers possess some familiarity with the subject content covered in such courses as advanced strength of materials, theories of elasticity and plates and shells, and differential equations.

GENERAL CONSIDERATIONS

Before and during World War II submarines[5] had been developed that could cruise on the surface at high speeds and had long ranges of operation under diesel power. They could dive when the occasion arose to attack ships and/or avoid detection, but they had to operate at greatly reduced speeds and for short periods of time on battery power while submerged until the need for air and battery-power replenishment required them to surface. Thus they and the post-World War II submarines shown in Figures 1 and 2 could be termed surface ships which were capable of submerging. The long and narrow hull configuration shown, permitted high surface speeds but offered only marginal underwater performance. Improvements in radar and sonar detection techniques almost immediately dictated that the underwater ships of the future be capable of prolonged submergence at greater depths and higher underwater speeds.

Therefore, in 1946 the Navy started a serious effort to develop a nuclear power plant for marine use; this effort culminated in the NAUTILUS (SSN-571) (shown in Figures 1 and 3) ---- which made it possible for man to cruise beneath the surface of the sea for extended periods of time. At the same time, a systematic study of the hydrodynamic characteristics of different hull shapes (similar to that conducted by the then NACA on airfoil sections) undertaken at the Model Basin led to the whale-like hull shape incorporated in the experimental submarine ALBACORE (AGSS-569) (shown in Figures 1 and 4), designed to give high underwater speeds.

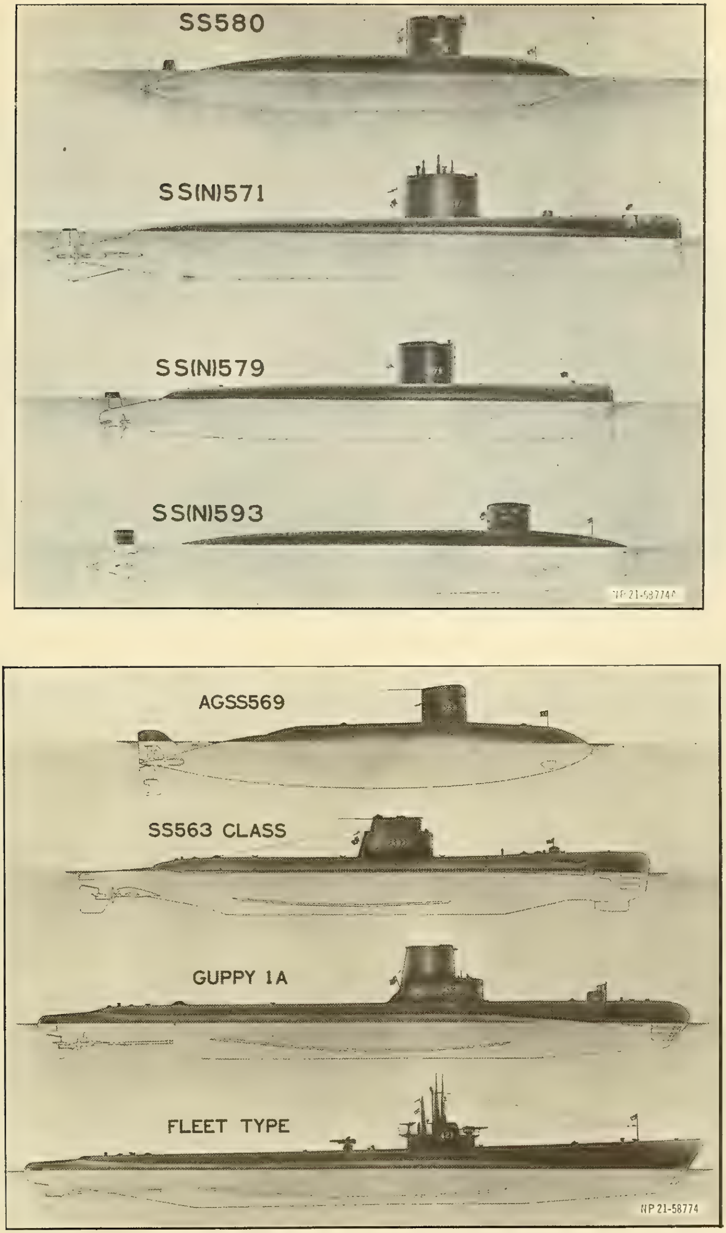

Figure 1 -- Outboard Profiles of Various Submarine Hulls

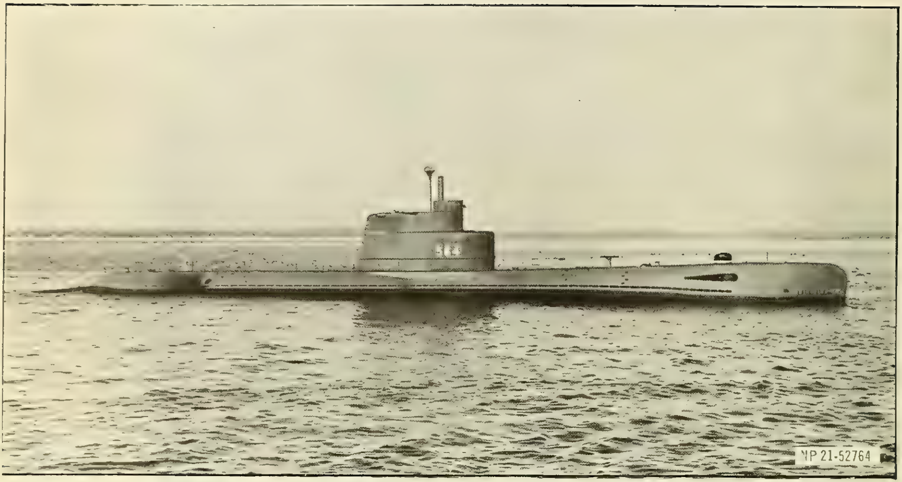

Figure 2 -- Post-World War II SS 563-Class Submarine

Figure 3 -- First Nuclear-Powered Submarine NAUTILUS

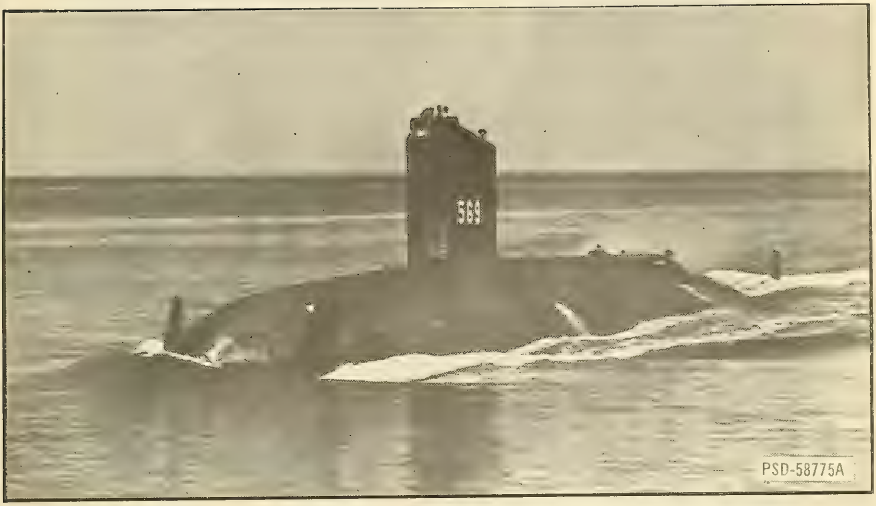

Figure 4 -- First Ideal-Shape Submarine ALBACORE

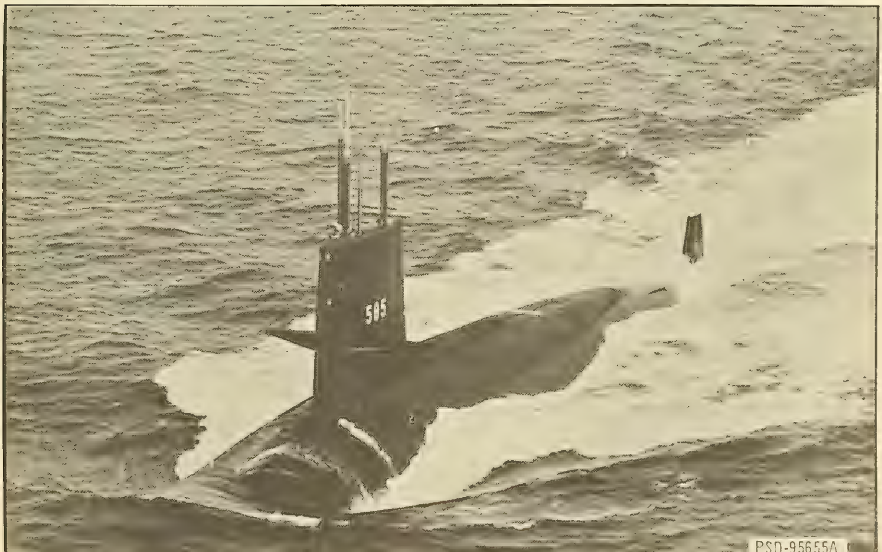

Figure 5 -- Nuclear-Powered, Ideal-Shape Submarine SKIPJACK

The marriage of the nuclear propulsion system of NAUTILUS with the hydrodynamically ideal hull shape of ALBACORE comprised a combination which could be exploited in producing a true submersible. Developments in hull structural materials also permitted operations at greater depths. These three major breakthroughs have led to the design and construction of two types of underwater ships: one, attack submarines of the SKIPJACK (SSN-585) class shown in Figure 5; and two, the strategic underwater deterrent system represented by the POLARIS missile-launching submarine shown in Figure 6. A more complete discussion of the various hull shapes may be found in Reference 2.

More recent developments, primarily in the area of structural analysis, and greater confidence in the use of the current hull structural material (HY-80), which is a high-strength, low-carbon, quenched and tempered martensitic steel with a nominal yield strength of 80,000 psi, have led to the design and initiation of construction of the DOLPHIN (AGSS-555), which is a deep-depth experimental submarine. Although this design represents a major increase in depth capability, it also signifies that practically every last bit of strength potential offered by this steel for hull structures of submersibles has been exploited.

In the strength analysis of submarine hulls, the most important element of structure, isolated from an otherwise complex structural system, is the ring-stiffened cylindrical shell. Conical transition sections

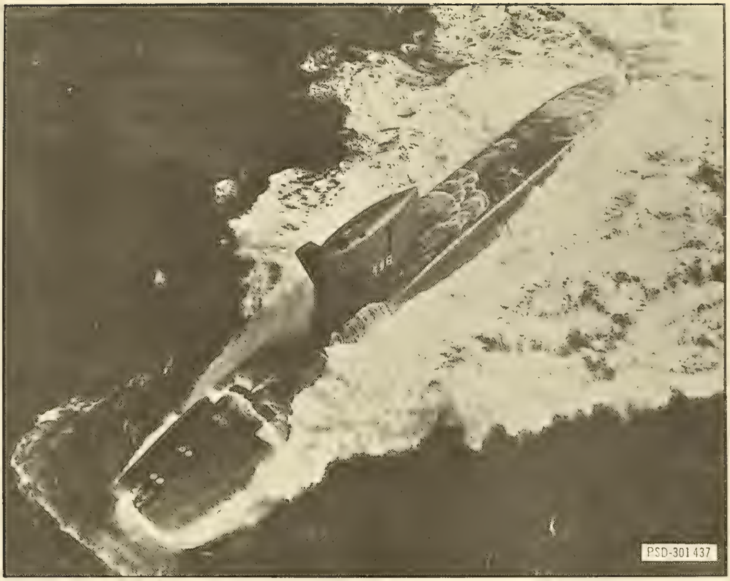

Figure 6 -- First Polaris-Firing Submarine GEORGE WASHINGTON

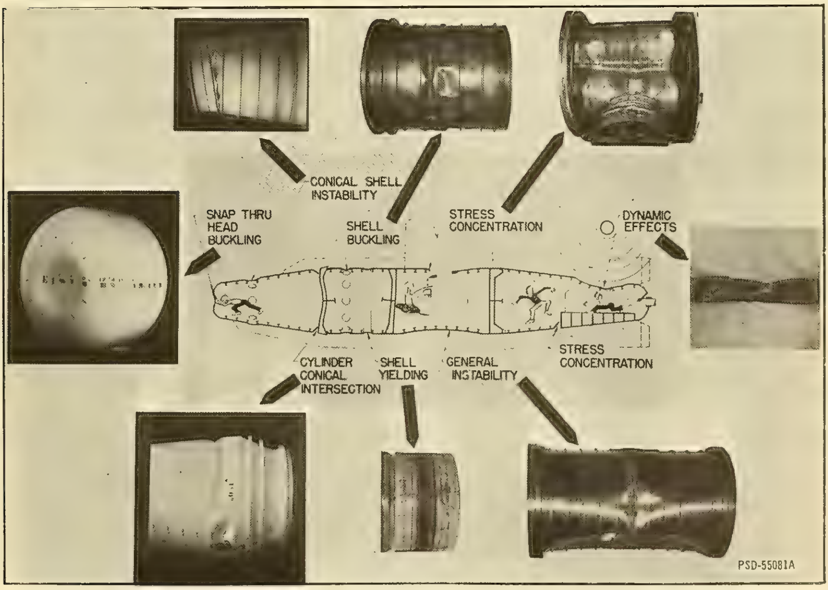

Figure 7 -- Possible Modes of Pressure Hull Failure

are used to some extent in joining cylindrical portions of different diameters, but they constitute a small part of the total hull structure so that some added strength could be allowed without a great sacrifice in weight. This is also true to some extent for the closure bulkheads used to terminate the cylindrical hulls. Some of the newer configurations which are being considered in conjunction with the higher-strength ferrous and non-ferrous materials for hulls of future deep-diving vehicles will be introduced in a later section of this presentation; however, a major part of our detailed discussions will be concerned with the elastic and inelastic behavior and the primary modes of collapse of ring-stiffened cylindrical pressure hulls under the action of uniform hydrostatic pressure.

Under the action of external hydrostatic pressure, failure of a ring-stiffened cylinder may be precipitated by any or a combination of three basic modes. The three distinct possible modes with which we will concern ourselves are indicated in Figure 7; they are:

1. Axisymmetric collapse of the shell between adjacent ring frames. This has been erroneously referred to by others [1,2] as yield failure of the shell, but in reality, it is a combination of yielding and axisymmetric buckling, or rather <u>inelastic axisymmetric shell instability between frames</u>. This mode is characterized by an accordion-type pleat which may or may not extend around the entire periphery, and which may or may not occur in more than one bay of the cylinder; a typical case is shown in Figure 8.

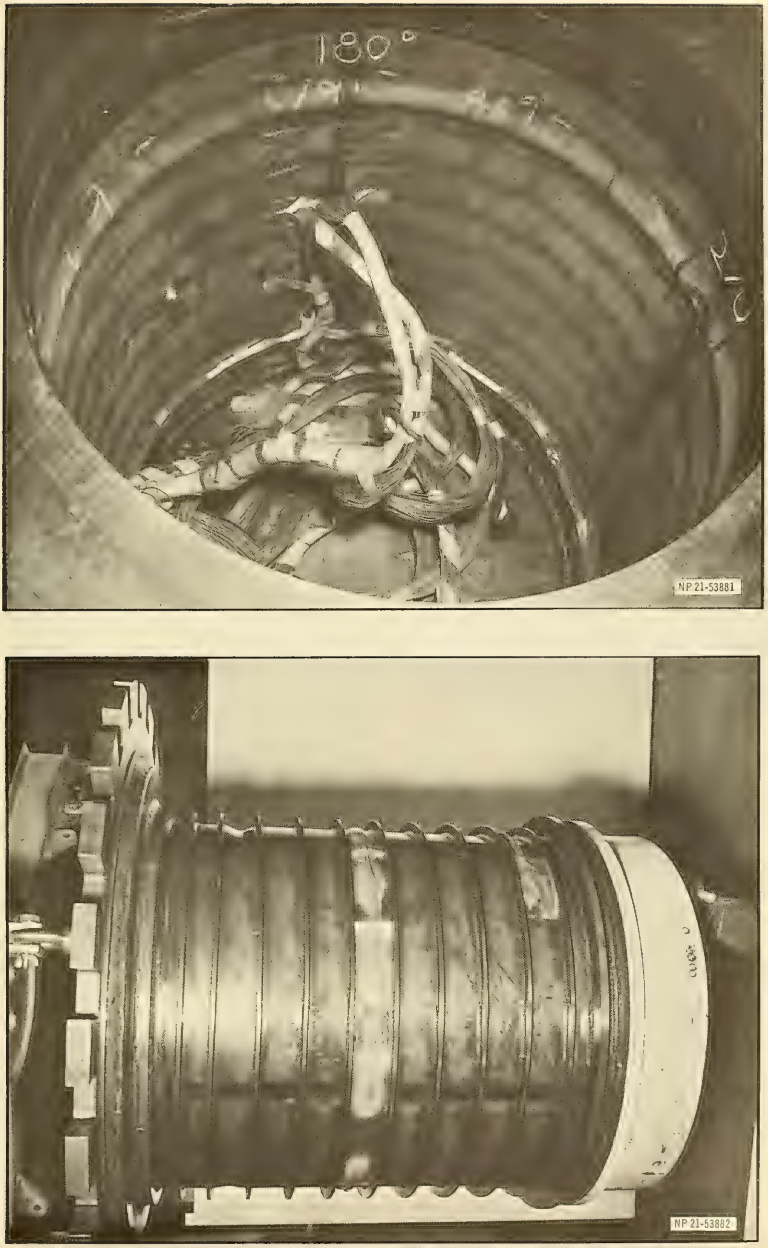

Figure 8 -- Typical Axisymmetric Collapse Mode

2. Nonaxisymmetric or asymmetric collapse of the shell between adjacent ring frames. This is usually referred to as shell or lobar buckling, and since in any good design, the shell is intended to be stressed beyond the elastic limit at design collapse, it is in reality, inelastic asymmetric shell instability between frames. This mode is characterized by inward-outward lobes which may or may not develop around the entire periphery, and which may or may not occur in more than one bay of the cylinder; a typical case is shown in Figure 9.

The basic difference between Modes 1 and 2, therefore, is in their axisymmetric and asymmetric patterns, respectively, and the question as to which may be the more critical of the two is governed by the thickness-radius ratio, frame spacing-radius ratio, frame cross-sectional area to shell cross-sectional area ratio of the geometric configuration, and the shape of the uniaxial stress-strain curve of the shell material. Imperfect circularity of the shell and deviations from straightness of the cylinder generators may also play some role in the interbay collapse of a ring-stiffened cylinder.

3. Overall asymmetric collapse of the shell and frames together, which may extend over the entire length of the ring-stiffened cylinder. In addition to the parameters mentioned above, the occurrence of this mode is strongly influenced by the moment of inertia of the ring frames and the ratio of the overall length to the radius of the cylinder. Imperfect circularity of the ring frames plays an important role here since it can precipi-

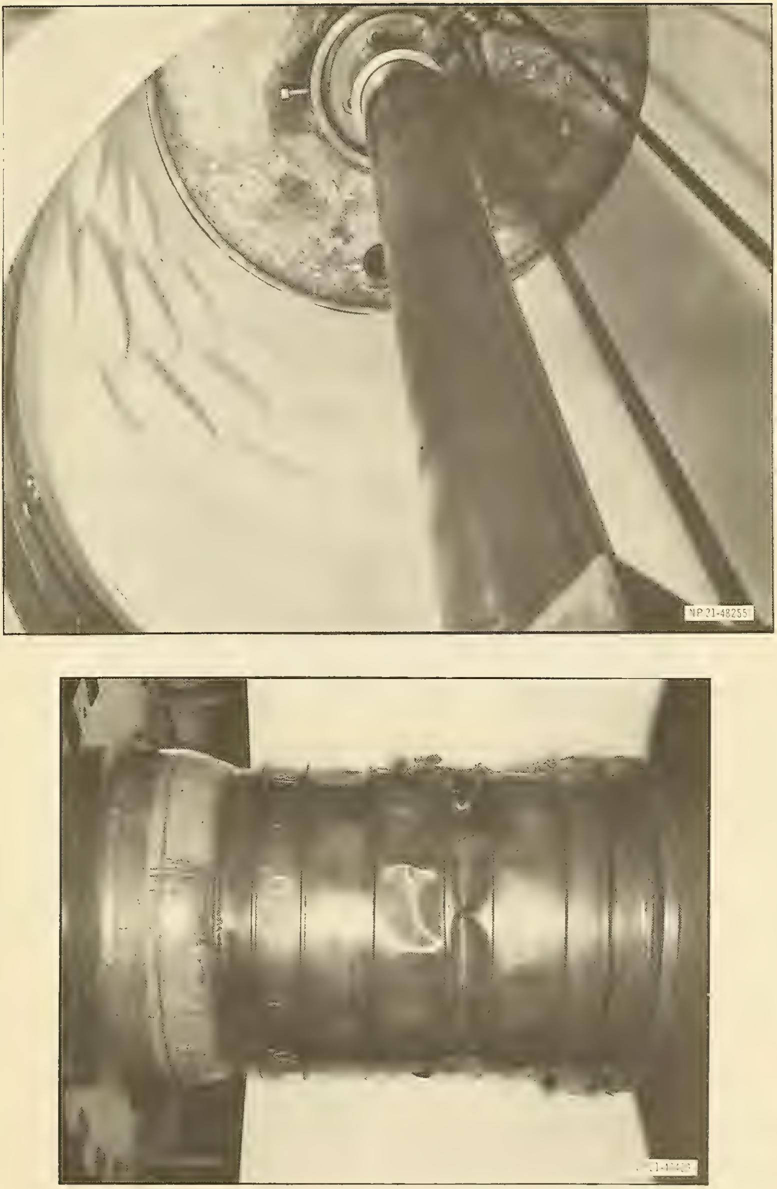

Figure 9 -- Typical Asymmetric Collapse Mode

tate a premature overall instability of long cylindrical hulls. Here, as in the case of the first two modes, instability of the elastic type is of interest only as a datum. It is inelastic overall instability which is of prime concern in designing an adequate hull; a typical case is shown in Figure 10.

Earlier authors[1,3] have offered the "one-hoss shay" concept, in which the best ratio of strength-to-weight is obtained when the hull structure is so designed that the three modes of collapse are expected to occur simultaneously, as the one for optimum design. However, this is a rather nebulous statement, and the approach even considered oversimplified if it is based on purely elastic considerations of instability because the "ignorance factors" or margins between the various modes would then merely be guesses, and may not necessarily be constant for the wide range of interest. Optimum design must be based on rational considerations of inelastic behavior, so that the "ignorance factors" can then be representative of the variability introduced by certain intangibles which are not easily considered in a theory.

To make structural problems of the type encountered in the analysis and design of pressure hulls for submersibles amenable to mathematical solution, the theoretician must invariably resort to idealizations and approximations of the actual physical conditions. In this way, the designer can only hope to realize upper and/or lower bounds on ultimate load-carrying capacity. Some of the intangibles which influence static strength and complicate the problem so that appropriate design formulas cannot be

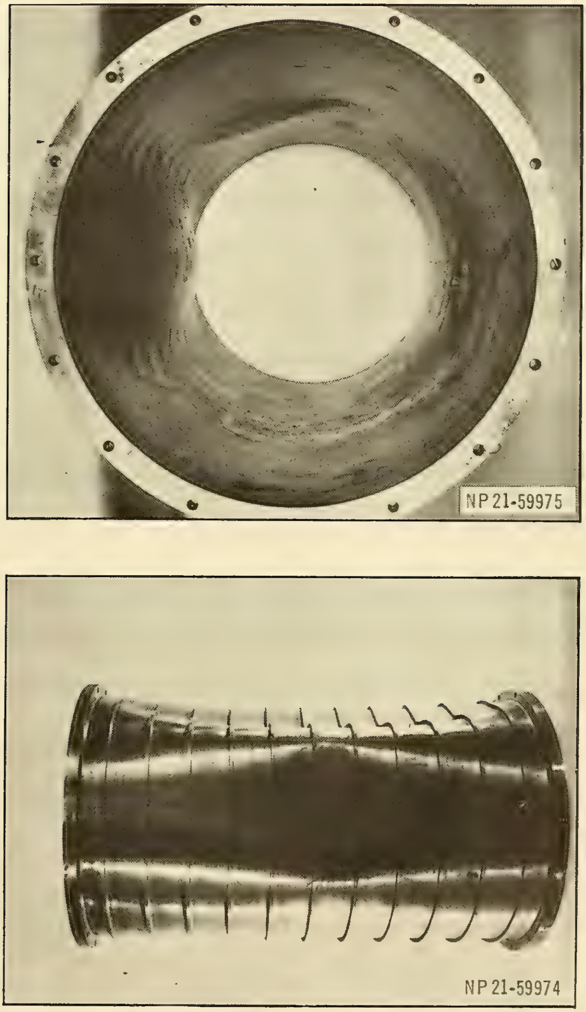

Figure 10 - Typical General-Instability Collapse Mode

derived on purely theoretical grounds are those inherent in the fabrication process itself since:

1.* Initial stresses, such as those due to rolling and welding of flat plating into the cylindrical form, can be of such magnitude so as to cause premature yielding and thus appreciably influence static collapse strength.

2.* Even the residual stresses which are introduced during the metal-fabrication process of rolling billets into flat plating and eventual heat treatment may also be of considerable magnitude.

3. Initial imperfect circularity (out-of-roundness) and axisymmetric longitudinal imperfections (deviations from straightness of the cylinder generators) introduce localized stresses which can lead to premature yielding and thus influence static collapse strength.

4. Even so-called isotropic and homogeneous materials (such as steel, and aluminum and titanium alloys) may not be so, to varying degrees depending on the plating thickness.

Finally, the elasto-plastic restraints afforded the shell plating by the stiffening ring frames are of a highly complicated nature due to varying degrees of localized yielding prior to collapse so that the actual boundary conditions may be different for each case, and would certainly deviate considerably from the ideal conditions (i.e., simple supports, clamped,

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* These considerations can have a great influence on the shape of the stress-strain curve for the material as it exists in the fabricated structure. They can lead to an effect commonly referred to as the Bauschinger Effect.

plastic hinges, etc.) assumed in any theory.

Therefore, in view of these complications, the development of satisfactory design criteria must first be predicated on rigorous mathematical theory with its concomitant idealizations, and then, empirical factors derived from test data can be introduced to "adjust" the theories to take account of those many variables which could not or were not considered. This is the procedure which has found extensive use in many fields for developing empirical design curves, and it is also proving very valuable to the naval architect in the design of pressure hulls. In what follows, consideration will first be given to the various theories which form the basis for predicting the elastic and inelastic behavior of ring-stiffened cylindrical pressure hulls, and then some indication as to how these theories can be used to predict collapse strength.

Following this there will be a general discussion of the strength-weight potential offered by a variety of ferrous and nonferrous metals, and even some of the new materials such as fiber-reinforced plastics which are being considered for future hull structures of deep-diving vehicles. This final section will also deal with some of the new and untried construction techniques and geometric configurations which are presently receiving detailed investigation.

AXISYMMETRIC BEHAVIOR OF A RING-STIFFENED CYLINDRICAL PRESSURE HULL

ELASTIC DEFORMATIONS AND STRESSES

In the past, many criteria for predicting the collapse strength of a ring-stiffened cylindrical pressure hull have been based on an accurate knowledge of the biaxial state of circumferential and longitudinal stress which results as a consequence of the uniform pressure loading. Although it recently has come to be appreciated that the axisymmetric mode of collapse is in reality a combined yielding and buckling phenomenon,* the prebuckling deformations and stresses are still a prime requisite to an analysis of this and the other (asymmetric) modes of instability.

The problem of determining the amount of external pressure that can be resisted by a cylindrical shell structure before failure is precipitated by axisymmetric yielding in the shell plating has been considered by many investigators, among them Von Sanden and Günther,[6] and Viterbo.[7] However, the criteria developed by these authors for the maximum bending and circumferential stresses in the shell plating and the peripheral load supported by the ring frames do not reflect properly the "beam-column" effect of the hydrostatic pressure on the structure. This effect is in reality the interaction between longitudinal bending and longitudinal compression as a consequence of the axial portion of the hydrostatic pressure loading. After close examination of these theories by the group at

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* Another way of looking at this is to consider it as axisymmetric buckling at a reduced modulus due to the stress state.

the Polytechnic Institute of Brooklyn, Salerno and Pulos around 1950 modified the earlier analyses to properly include the complete effect of the pressure load. What has come to be accepted as the most up-to-date solution and discussion of this problem is given in Reference 8. For our purposes here, it suffices to give the governing differential equation and its general solution, the boundary conditions invoked, and finally, the expressions for the shell and frame stresses which are later used in formulating some of the accepted and often used collapse criteria.

The differential equation governing the axisymmetric elastic deformations of a thin-walled circular cylindrical shell of finite length and under the action of uniform external hydrostatic pressure (see Figure 11) is

$$D \frac{d^4 w}{dx^4} + \frac{pR}{2} \frac{d^2 w}{dx^2} + \frac{Eh}{R^2} w = -p(1-\nu/2) \tag{1}$$

where the necessary nomenclature and definitions used here and in the equations to follow are indicated in the Notation. A derivation of Equation (1) may be found in either References 8 or 9.

The term $\frac{pR}{2} \frac{d^2 w}{dx^2}$ which renders the solution of Equation (1) to be a nonlinear function of the pressure is the "beam-column effect" which was not considered in the analyses of References 6 and 7. The importance of this term is further emphasized by the fact that it is the necessary ingredient for extracting a criterion for axisymmetric elastic buckling of a

cylindrical shell supported by ring frames of finite stiffness; this is discussed more fully in Reference 8.

It can be shown that the general solution of Equation (1), which is a linear, ordinary differential equation with constant coefficients, can be written in the form

$$w(x) = A \sinh \lambda_1 x + B \cosh \lambda_1 x + C \sinh \lambda_3 x + F \cosh \lambda_3 x - \frac{pR^2}{Eh}(1 - \nu/2) \qquad (2)$$

The characteristic roots λ_1 and λ_3 of the fourth degree auxiliary equation which results when $w(x) \sim e^{\lambda x}$ is substituted into Equation (1) are given by

$$\lambda_1 ; \lambda_3 = \sqrt{2}\,\frac{\theta}{L}\left\{-\left(\frac{p}{p^*}\right) \pm \left[\left(\frac{p}{p^*}\right)^2 - 1\right]^{1/2}\right\}^{1/2} \qquad (3)$$

where

$$\theta = \sqrt[4]{3(1-\nu^2)}\,\frac{L}{\sqrt{Rh}} \qquad (4)$$

and

$$p^* = \frac{2E(h/R)^2}{\sqrt{3(1-\nu^2)}} \qquad (5)$$

is the critical load for the axisymmetric elastic buckling of an unstiffened cylindrical shell under the action of uniform axial pressure (see Reference 9).

An examination of the roots λ_1 and λ_3 reveals that four possibilities for the ratio p/p^* exist and that they all influence the nature of Equation (2). These are

$$\frac{p}{p^*} \lesseqgtr 1.0; \frac{p}{p^*} = 0 \tag{6}$$

The case of greatest importance and the only one which will be considered here is that defined by $p/p^* < 1.0$. A complete discussion of all possible solutions defined by Equation (6) is given in Reference 8.

The integration constants A, B, C, and F appearing in Equation (2) are evaluated from the following boundary conditions:

(a) Evenness of the function w(x) dictates A = C = 0

(b) Zero slope at the ring frames: $\frac{dw}{dx} = 0$ at $x = \frac{L}{2}$

(c) Compatability of the shell and frame radial deflections:

$$\frac{E}{R^2}(A_{eff} + bh)w = 2D\frac{d^3w}{dx^3} - pR\frac{dw}{dx} - pb(1-\nu/2) \quad \text{at } x = \frac{L}{2}$$

The boundary condition (c) results from the fact that the total load supported by a ring frame per unit circumferential length is

$$|Q^*| = 2Q_o + pb(1-\nu/2) \tag{7}$$

and the transverse shear force Q_o is that transmitted by the shell to the frame at their juncture and is given by

$$Q_o \equiv \left[Q_x = -D\frac{d^3w}{dx^3} + \frac{pR}{2}\frac{dw}{dx} \right]_{x = \frac{L}{2}} \tag{8}$$

Also, the effective frame cross-sectional area $A_{eff.}$ appearing in boundary condition (c) is defined by the following for internal and external framing, respectively:

$$A_{eff} = A_f\left(\frac{R}{R_{cg}}\right)$$

$$A_{eff} = A_f\left(\frac{R}{R_{cg}}\right)^2 \tag{9}$$

where R is the radius to the median surface of the shell and R_{cg} is the radius to the centroid of the frame cross section.

The total longitudinal stress σ_X in the shell is given as the sum of the longitudinal bending component σ_{xb} plus the longitudinal membrane component σ_{xM}, i.e.,

$$\sigma_X = \pm \frac{Eh}{2(1-\nu^2)} \frac{d^2w}{dx^2} - \frac{pR}{2h} \tag{10}$$

so that enforcing boundary condition (a) in Equation (2) results, in

$$\sigma_{X_i^o} = \pm \frac{Eh}{2(1-\nu^2)}\left[B\lambda_1^2\cosh\lambda_1 x + F\lambda_3^2\cosh\lambda_3 x\right] - \frac{pR}{2h} \tag{11}$$

The total circumferential stress σ_Φ in the shell is given as the sum of the circumferential bending component $\nu\sigma_{xb}$ plus the circumferential membrane component $\sigma_{\phi M}$, i.e.,

$$\sigma_\Phi = \pm\frac{\nu Eh}{2(1-\nu^2)}\frac{d^2w}{dx^2} + \frac{E}{R}w - \nu\frac{pR}{2h} \tag{12}$$

so that

$$\sigma_{\Phi_{i}^{o}} = -\frac{pR}{h} + \left[\frac{E}{R} \pm \frac{\nu E h \lambda_1^2}{2(1-\nu^2)}\right] B \cosh \lambda_1 x + \left[\frac{E}{R} \pm \frac{\nu E h \lambda_3^2}{2(1-\nu^2)}\right] F \cosh \lambda_3 x \qquad (13)$$

where the subscripts o and i in Equations (11) and (13) refer to the outer and inner fibers of the shell plating in conjunction with the plus and minus signs, respectively.

The total radial ring-load Q^* is found from Equations (7), (8) and (2) to be

$$Q^* = \frac{Eh^3}{6(1-\nu^2)}\left[B \lambda_1^3 \sinh \lambda_1 \frac{L}{2} + F \lambda_3^3 \sinh \lambda_3 \frac{L}{2}\right] - pb(1-\nu/2) \qquad (14)$$

When boundary conditions (b) and (c) are invoked, the integration constants B and F are determined and the following expressions for the more important shell stresses are derived in terms of the convenient notation of Reference 10: <u>at midbay between adjacent ring frames</u> (i.e., at x = o)

$$\frac{\sigma_{X_{i}^{o}m}}{\sigma_u} = \frac{1}{2} \pm \frac{\sigma_{xbm}}{\sigma_u} \qquad (15)$$

$$\frac{\sigma_{\Phi_{i}^{o}m}}{\sigma_u} = 1 - \left(1 - \frac{\sigma_{\phi Mf}}{\sigma_u}\right) F_2 \pm \nu \frac{\sigma_{xbm}}{\sigma_u} \qquad (16)$$

at a ring frame (i.e., at $x = \frac{L}{2}$)

$$\frac{\sigma_{x o_{f_i}}}{\sigma_u} = \frac{1}{2} \pm \left(1 - \frac{\sigma_{\phi Mf}}{\sigma_u}\right) \sqrt{\frac{0.91}{1-\nu^2}}\, F_3 \tag{17}$$

$$\frac{\sigma_{\Phi o_{f_i}}}{\sigma_u} = 1 - \left(1 - \frac{\sigma_{\phi Mf}}{\sigma_u}\right) \pm \nu \left(1 - \frac{\sigma_{\phi Mf}}{\sigma_u}\right) \sqrt{\frac{0.91}{1-\nu^2}}\, F_3 \tag{18}$$

and where

$\sigma_u = -\dfrac{pR}{h}$ is the circumferential stress in an unstiffened cylinder of infinite length,

σ_{xbm} is the longitudinal bending stress in the shell at midbay between adjacent ring frames, and

$\sigma_{\phi\, Mf}$ is the circumferential membrane stress in the shell at a ring frame.

These latter two stresses are given by

$$\frac{\sigma_{xbm}}{\sigma_u} = \sqrt{\frac{0.91}{1-\nu^2}} \left[\frac{(1-\nu/2)\alpha}{\alpha + \beta + (1-\beta)F_1}\right] F_4 \tag{19}$$

$$\left(1 - \frac{\sigma_{\phi\, Mf}}{\sigma_u}\right) = \frac{(1-\nu/2)\alpha}{\alpha + \beta + (1-\beta)F_1} \tag{20}$$

and where:

$$\alpha = \frac{A_{eff}}{L_f h}\ ;\ \beta = \frac{b}{L_f}$$

In Reference 11, Lunchick and Short modify the theory of Reference 8 to include the effect of initial axisymmetric deviations from straightness of the cylinder generators. If it is assumed that this deviation possesses constant curvature between adjacent ring frames so that it can be expressed analytically by a second-degree parabola and assumes a maximum amplitude Δ (+ inward), then the stress expressions, i.e., Equations (15) through (18), respectively, become in the present notation

$$\frac{\sigma_{x_i^{o}m}}{\sigma_u} = \frac{1}{2} \pm \sqrt{\frac{0.91}{1-\nu^2}}\left[1 + \frac{4R\Delta}{L^2(1-\nu/2)}\left(\frac{\alpha+1}{\alpha}\right)\right] aF_4 \qquad (15')$$

$$\frac{\sigma_{\Phi_i^{o}m}}{\sigma_u} = 1 - aF_2 + \frac{4R\Delta}{L^2}\left[1 - \frac{(\alpha+1)F_2}{\alpha+\beta+(1-\beta)F_1}\right] + $$

$$\pm\nu\sqrt{\frac{0.91}{1-\nu^2}}\left[1 + \frac{4R\Delta}{L^2(1-\nu/2)}\left(\frac{\alpha+1}{\alpha}\right)\right] aF_4 \qquad (16')$$

$$\frac{\sigma_{x_i^{o}f}}{\sigma_u} = \frac{1}{2} \pm \sqrt{\frac{0.91}{1-\nu^2}}\left[1 + \frac{4R\Delta}{L^2(1-\nu/2)}\left(\frac{\alpha+1}{\alpha}\right)\right] aF_3 \qquad (17')$$

$$\frac{\sigma_{\Phi_i^{o}f}}{\sigma_u} = 1 - a\left[1 + \frac{4R\Delta}{L^2(1-\nu/2)}\,\frac{(1-\beta)(1-F_1)}{\alpha}\right] + $$

$$\pm\nu\sqrt{\frac{0.91}{1-\nu^2}}\left[1 + \frac{4R\Delta}{L^2(1-\nu/2)}\left(\frac{\alpha+1}{\alpha}\right)\right] aF_3 \qquad (18')$$

in which the short-hand notation

$$a \equiv \frac{(1-\nu/2)\alpha}{\alpha+\beta+(1-\beta)F_1}$$

has been introduced. For the case of no eccentricity, i.e., $\Delta = 0$, Equations (15') through (18') reduced to the corresponding stress expressions given previously. Also, in the same convenient notation, the following expression for the total radial load acting on a ring frame per unit circumferential length is obtained:

$$Q^* = -pb(1-\nu/2)\left\{1 + \frac{(1-\beta)\frac{\alpha}{\beta}F_1}{\alpha+\beta+(1-\beta)F_1}\right\} \qquad (21)$$

which corresponds to the case of zero initial axisymmetric eccentricity.

In the case of a ring-stiffened cylinder under some loading, such as hydrostatic pressure which is of interest in the case here, a portion of the deformed shell between stiffeners will act effectively with each ring frame to resist direct stress and bending moment caused by the interaction between the shell and the frames. A knowledge of this "effective width" is of particular interest in a study of the buckling strength of the ring itself and in the elastic and inelastic general-instability analyses of the entire stiffened cylinder (this problem is considered in a later section). It is also important in calculating the stresses in the frame flanges of imperfectly circular cylindrical shell structures.

In Reference 8, Pulos and Salerno discuss the many "effective width" formulas developed by earlier investigators, and they present a formal derivation of a new formula to include the "beam-column" effect. Details

of this may be found in Reference 8; however it is of interest here to give the end result which can be expressed by the following convenient formula:

$$L_e = LF_1 + b \tag{22}$$

In Equations (15) through (22) the following F functions (graphical solutions for which were first developed by Krenzke and Short in Reference 10) have been introduced for ease of calculation:

$$F_1 = \left(\frac{4}{\theta}\right)\left[\cosh^2 \eta_1\theta - \cos^2 \eta_2\theta\right] / \mathcal{D} \tag{23}$$

$$F_2 = \left[\frac{\cosh \eta_1\theta \sin \eta_2\theta}{\eta_2} + \frac{\sinh \eta_1\theta \cos \eta_2\theta}{\eta_1}\right] / \mathcal{D} \tag{24}$$

$$F_3 = \sqrt{\frac{3}{0.91}}\left[-\frac{\cosh \eta_1\theta \sinh \eta_1\theta}{\eta_1} + \frac{\cos \eta_2\theta \sin \eta_2\theta}{\eta_2}\right] / \mathcal{D} \tag{25}$$

$$F_4 = \sqrt{\frac{3}{0.91}}\left[\frac{\cosh\eta_1\theta\sin\eta_2\theta}{\eta_2} - \frac{\sinh\eta_1\theta\cos\eta_2\theta}{\eta_1}\right] / \mathcal{D} \tag{26}$$

where:

$$D = \left[\frac{\cosh\eta_1\theta\sinh\eta_1\theta}{\eta_1} + \frac{\cos\eta_2\theta\sin\eta_2\theta}{\eta_2}\right]$$

$$\eta_1 = \frac{1}{2}\sqrt{1-\Upsilon}\,; \quad \eta_2 = \frac{1}{2}\sqrt{1+\Upsilon}$$

$$\Upsilon \equiv \frac{p}{p^*} = \frac{p}{2E}\sqrt{3(1-\nu^2)}\left(\frac{R}{h}\right)^2 \tag{27}$$

$$\theta = \sqrt[4]{3(1-\nu^2)}\,\frac{L}{\sqrt{Rh}}$$

Curves of the functions F_1, F_2, F_3, and F_4 may be found in either of References 8 or 10.

The elastic analysis developed in Reference 8 is intended for the determination of the deformations and stresses in a typical bay of a pressurized ring-stiffened cylinder composed of many identical bays as shown in Figure 11. This longitudinal identity and symmetry between adjacent bays is disturbed by the introduction of rigid bulkheads, intermediate deep frames, cone and sphere-cylinder junctures, and other contiguous structure which goes to make up the pressure hull of a submersible. In these more complicated configurations, a more general analysis of the axisymmetric behavior is needed.

Short and Bart have given a general analysis for determining the stresses in stiffened cylindrical shells near structural discontinuities.[12] The formulation includes the possibility that the shell thickness may differ in adjacent bays, the stiffness properties of adjacent ring-frames may be different, and the spacing between ring frames may vary along the length of the cylindrical compartment. The general equations developed by these investigators are given in the form of frame and shell matrices to better identify the stiffness and response of each element and to facilitate numerical calculations. This form of the solution lends itself very conveniently to high-speed digital computers and also permits immediate identification of those geometric and material properties which can be varied to produce desirable changes in the static response. All the

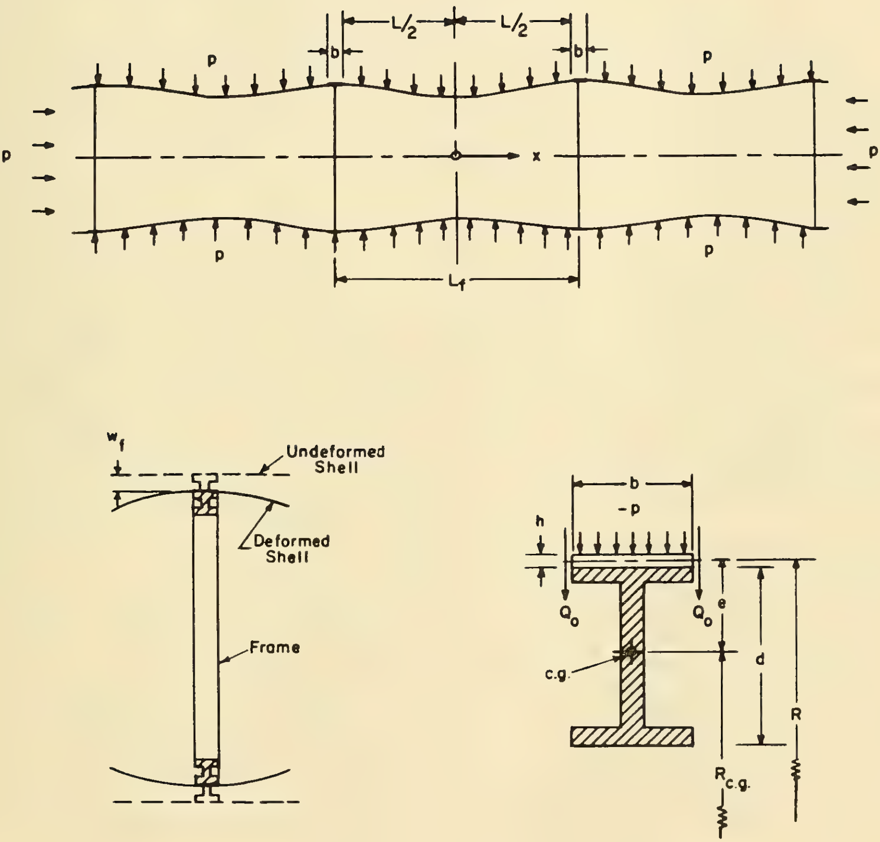


Figure 11 - Symmetrically Loaded Cylindrical Shell with Equally Spaced Reinforcing Ring Frames

necessary equations and details of the formulation may be found in Reference 12; they are not given here because of their length and rather formidable nature. However, it is worth mentioning here that extensive experimental evaluation has been obtained of the Short-Bart analysis. For example, Keefe and Overby[13] present the results of structural model tests undertaken to check the "end-bay" theory Reference 12. Also, Keefe and Short[14] present a method for eliminating the effect of end conditions on the static collapse strength of stiffened cylindrical pressure hulls and give experimental verification of the suggested procedure.

Another special problem of interest to pressure hull designers, and one worth mentioning here, is that concerned with the discontinuity stresses which arise at the juncture of axisymmetric shells possessing dissimilar meridional shape. Raetz and Pulos[15] present an analysis for determining the elastic deformations occurring at either cone-cylinder or cone-cone junctures and discuss several other analyses developed by earlier investigators, notably Wenk and Taylor.

Conical transition sections are used rather extensively to join cylindrical hull components of different diameter, and not only is the problem of the edge effects on static collapse strength of the "weakened" bays of concern, but also, the occurrence of high, localized longitudinal stresses in these juncture regions is of great concern from the point of view of low-cycle fatigue in way of welded joints. Raetz[16] discusses this problem and suggests the use of flexible, tapered ring-segments at these

junctures to reduce the high longitudinal stresses; he also presents an analysis for determining the elastic behavior of these structural elements and gives results to indicate the degree of reduction which can be realized in these high stresses.

FAILURE CRITERIA FOR AXISYMMETRIC COLLAPSE PRECIPITATED BY YIELDING

We will now consider the question of how the biaxial stresses (defined by Equations (15) through (20)) in a pressurized ring-stiffened cylinder can combine to produce axisymmetric collapse precipitated by yielding of the shell plating. Although these stresses are based on equilibrium considerations only and do not reflect any buckling state, they can and do predict good results when used in conjunction with appropriate theories of failure even though, strictly speaking, axisymmetric collapse is associated with an instability phenomenon. Formulas for predicting axisymmetric collapse precipitated by yielding based on various theories of failure are summarized in Reference 8.

The simplest formula devised for the design of pressure vessels is the so-called "boiler formula". This formula may have some merit in predicting the bursting strength of internally pressurized unstiffened cylindrical tubes, but it is unsatisfactory (for other than comparative purposes) in the design of pressure hulls in which instability and the influence of reinforcing ring frames play a dominant role.

Equation (28) gives the pressure at which the circumferential membrane stress in an unstiffened cylinder of mean radius R and

thickness h just reached the yield stress σ_y of the material.

$$p_b = \sigma_y h/R \tag{28}$$

Equation (28) does not reflect in any way the strengthening effect of the transverse ring frames on the average circumferential stress. However, an estimate of this effect can be found by assuming that the cross-sectional area of the frames is spread out and its orthotropic stiffness effect is "felt" in the form of a thicker unstiffened cylindrical shell. This requires that the actual thickness h in Equation (28) be replaced by

$$h + \frac{A_{eff}}{L_f}$$

so that we now get the following modified boiler formula:

$$p_{c1} = \sigma_y h(1+ A_{eff}/L_f h)/R \tag{29}$$

From the theory of Salerno and Pulos outlined earlier, the maximum stresses occur in the circumferential direction on the outside surface of the shell plating midway between adjacent ring frames, and in the longitudinal direction on the inside surface of the shell plating at a frame; these stresses can be determined from Equations (16) and (17), respectively. Which of the two stresses is the larger depends upon the geometry of the cylindrical shell and the reinforcing ring frames, but in most cases of interest, it turns out that $\sigma_{x_i f} > \sigma_{\Phi o m}$. However, extensive Model Basin tests have shown that the stress $\sigma_{\Phi o m}$ is determinative in precipitating axisymmetric collapse. Application of the maximum principal stress theory of Rankine[17] to this stress, i.e.,

$$\sigma_{\Phi o m} \equiv \sigma_y \tag{30}$$

leads to the following expression for the pressure at which yielding begins on the outside fiber of the shell plating midway between adjacent ring frames:

$$p_{c3} = \frac{\sigma_y h/R}{1-a\left[F_2-\nu F_4\sqrt{\frac{0.91}{1-\nu^2}}\right]} \tag{31}$$

where $a \equiv \dfrac{(1-\nu/2)\alpha}{\alpha+\beta+(1-\beta)F_1}$.

If the uniaxial criterion of Rankine is applied to the circumferential membrane (midfiber) stress $\sigma_{\phi M}$, i.e., Equation (16) with σ_{xbm} set equal to zero, the following expression for the pressure at which yielding has penetrated through the plating thickness is obtained:

$$p_{c5} = \frac{\sigma_y h/R}{1-aF_2} \tag{32}$$

In deriving the expressions for p_{c1}, p_{c3}, and p_{c5}, it has tacitly been assumed that a uniaxial state of stress exists in the shell plating whereas, in reality, a biaxial state exists. More realistic criteria for axisymmetric collapse can be derived from the energy-of-distortion theory[17] which grew out of the analytical work of Huber, Von Mises, and Hencky. Since the octahedral shear-stress theory gives the same results as the energy-of-distortion theory and permits the use of a more familiar quantity, such as stress, the former theory will be used in what follows. For a biaxial state of stress at midbay, i.e., midway between adjacent ring frames, defined by the principal stresses σ_{xm} and $\sigma_{\Phi m}$, the octahedral shear-stress is

given by

$$\tau_G = \frac{1}{3}\left[(\sigma_{Xm} - \sigma_{\Phi m})^2 + \sigma_{Xm}^2 + \sigma_{\Phi m}^2\right]^{1/2} \tag{33}$$

However, since according to this theory inelastic action at any point in a body under any combination of stresses begins only when the octahedral shear-stress τ_G becomes equal to $(\sqrt{2}/3)\sigma_y$, then Equation (33) leads to the following:

$$\left[\sigma_{Xm}^2 + \sigma_{\Phi m}^2 - \sigma_{Xm}\sigma_{\Phi m}\right]^{1/2} \equiv \sigma_y \tag{34}$$

Essentially two distinct criteria can be derived from Equation (34) depending upon whether the outer-fiber stresses or mid-fiber (membrane) stresses at midbay are used. For yielding on the outer surface of the shell plating, when the appropriate stresses $\sigma_{X^o m}$ and $\sigma_{\Phi^o m}$ from Equations (15) and (16), respectively, are substituted into Equation (34), the following criterion is obtained:

$$p_{c6} = \frac{\sigma_y h/R}{\left\{\frac{3}{4} + a^2\left[F_2^2 + F_2F_4(1-2\nu)\sqrt{\frac{0.91}{1-\nu^2}} + F_4^2(1-\nu+\nu^2)\left(\frac{0.91}{1-\nu^2}\right)\right] - \frac{3}{2}a\left[F_2 - \nu F_4\sqrt{\frac{0.91}{1-\nu^2}}\right]\right\}^{1/2}} \tag{35}$$

If it is assumed that axisymmetric collapse is precipitated by the yield zone having penetrated through the shell thickness, when the appropriate stresses σ_{Xm} and $\sigma_{\Phi m}$ from Equations (15) and (16) with σ_{xbm} set equal to zero therein, respectively, are substituted into Equation (34), the following criterion is obtained:

$$p_{c7} = \frac{\sigma_y h/R}{\left[\frac{3}{4}+a^2F_2^2-\frac{3}{2}aF_2\right]^{1/2}} \tag{36}$$

Lunchick[18] working at the Model Basin derived another criterion of failure for predicting axisymmetric collapse which is based on the plastic-hinge concept. He made use of the Hencky-Huber-Von Mises theory of yielding, i.e., Equation (34), and allowed for the plastic reserve strength after yielding begins in the shell plating at midbay. Lunchick developed a formula for the pressure at which a complete plastic hinge has formed at midbay. Since the combined stress gradients at the frame locations are steeper than those at midbay so that complete plastic hinges form much earlier at the frames, this TMB plastic-hinge theory, in reality, gives the pressure at which the shell fails as a three-hinge mechanism. Although this mechanism is not physically possible in the case of cylindrical shells as it is for beams, it does lead to predictions of a collapse pressure (p_{c8}) which agree well with experiment in certain ranges of geometry. A complete discussion of this theory together with some comparisons to experimental data may be found in Reference 18. For our purposes here, it suffices to give the salient results which can be used for computation.

It can be shown that the ratio of circumferential bending stress to circumferential membrane stress and the ratio of longitudinal membrane stress to circumferential membrane stress (all stresses considered at midbay) can be expressed in the convenient notation of Reference 8 and

that adopted here as follows:

$$\sigma_{\phi bm}/\sigma_{\Phi M} = \frac{\nu a F_4 \sqrt{\dfrac{0.91}{1-\nu^2}}}{1-aF_2} \tag{37}$$

$$\sigma_{XM}/\sigma_{\Phi M} = \frac{0.5}{1-aF_2} \tag{38}$$

In Figure 12, the pressure ratio p_{c8}/p_{c6} has been plotted as a function of the stress ratios, Equations (37) and (38). Thus, Equations (37) and (38) can be used in conjunction with these curves and Equation (35) for p_{c6} to determine values of the plastic-hinge pressure p_{c8} for different geometries.

The formulas for predicting axisymmetric collapse precipitated by yielding, and given in this section, represent explicit expressions for collapse pressure only for the special case of zero "beam column" effect, i.e., $\gamma = 0$, since in this case only are the F functions given by Equations (23) through (26) independent of pressure. For the general case in which $\gamma \neq 0$, the stresses become nonlinear functions of the pressure, and Equations (31), (32), (35), and (36) are transcendental in the pressure. However, a numerical iteration procedure can be used in which the collapse pressures p_{c3}, p_{c5}, etc. are first calculated for $\gamma = 0$, and these values are then used as the first approximation in the last of Equations (27) to determine a value of γ. Then, with this value of γ in each corresponding case, new values of the pressures p_{c3}, p_{c5}, ... etc. can be found. This

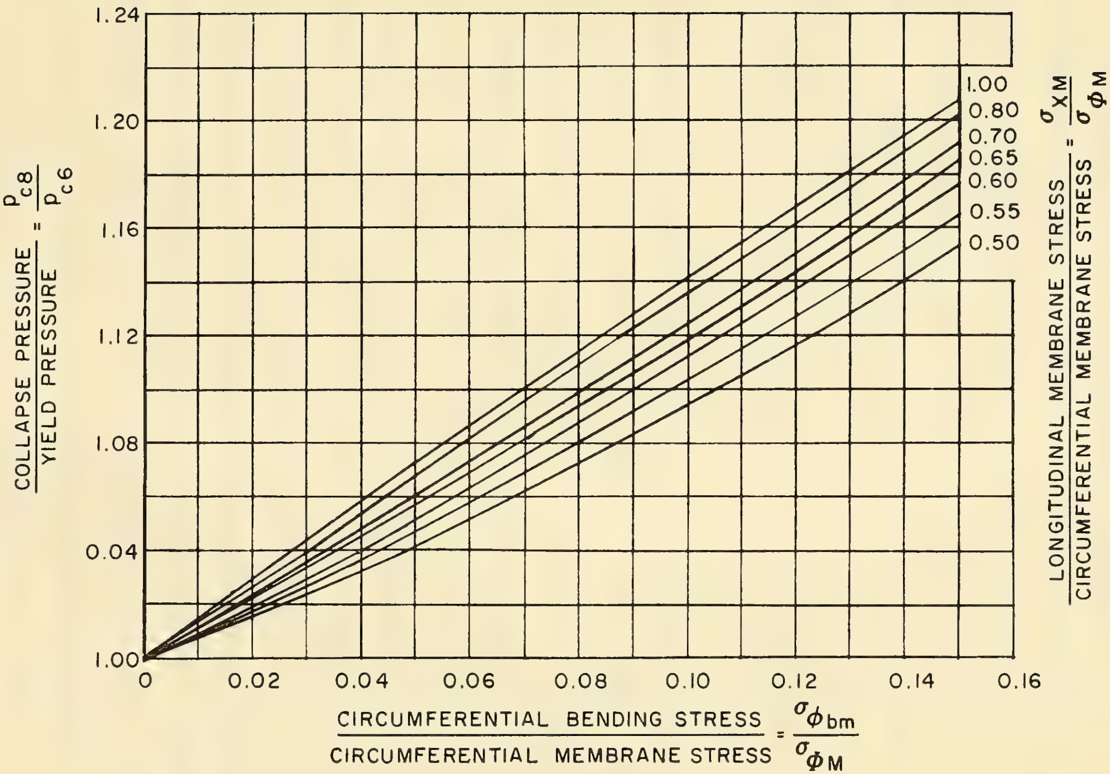


Figure 12 - Curves for Computing Plastic-Hinge Collapse Pressure p_{c8}

process can be repeated again, but probably one or two, or at most, three iterations are required to obtain satisfactory convergence.

One important point to be made is that the collapse criteria reflected by the expressions for p_{c3}, p_{c5}, p_{c6}, p_{c7}, and p_{c8} are inherently based on a material which exhibits elastic-perfectly plastic characteristics, so that any effects of strain-hardening are neglected. The assumption is made that once the yield strength σ_y is reached, at the critically stressed mid-bay locations according to each of the respective criteria, collapse then occurs.

Another solution for axisymmetric yield collapse of a ring-stiffened pressurized cylinder in the presence of the "beam-column" effect was developed by Kempner and Salerno.[19] Their elastic analysis was predicated on the Hencky-Huber-Von Mises failure criterion as applied to the critically stressed fiber at the midbay location. For the usual range of geometries the yield strength of the material is first reached at the ring frames due to the high longitudinal stress on the inner fiber. The authors then considered that for loading beyond this initial yielding pressure, a full plastic-hinge exists at the stiffening rings. Thus, the additional pressure which the shell can support is calculated on the assumption that the shell plating is "hinged" to the ring frames. The maximum additional pressure which the shell can support is then based on applying the aforementioned failure criterion to the total stress at the midbay location.

AXISYMMETRIC ELASTIC BUCKLING BETWEEN RING FRAMES

In reference 8, Salerno and Pulos derive the following criterion which represents an exact solution to the axisymmetric elastic buckling problem of a thin cylindrical shell reinforced by ring frames of finite rigidity and loaded by hydrostatic pressure:

$$\alpha + \beta + (1-\beta)F_5 = 0 \tag{39}$$

where

$$F_5 = \frac{4}{\theta}\left[\frac{\cos^2\eta_1'\theta - \cos^2\eta_2\theta}{\dfrac{\cos\eta_1'\theta\sin\eta_1'\theta}{\eta_1'} + \dfrac{\cos\eta_2\theta\sin\eta_2\theta}{\eta_2}}\right] \tag{40}$$

and

$$\eta_1' = \frac{1}{2}\sqrt{\gamma - 1}$$

$$\eta_2 = \frac{1}{2}\sqrt{\gamma + 1} \tag{41}$$

$$\gamma = \frac{p}{2E}\sqrt{3(1-\nu^2)}\left(\frac{R}{h}\right)^2$$

Equation (39) is a transcendental equation to be solved for the critical pressure $p = p_{cr}$ for a given shell and ring-frame geometry defined by the nondimensional parameters α, β, and θ. A graphical representation of Equation (40) is given in Reference 8 to facilitate the iteration calculations required to find the value of p which satisfies the buckling criterion, Equation (39).

In small-displacement theory, the condition for instability of a

structure may be obtained by requiring that the displacements become infinitely large at buckling. It is on this basis that Salerno and Pulos arrived at the criterion of Equation (39). Since Von Sanden and Günther,[6] and later Viterbo,[7] omitted the beam-column term in the basic differential equation, Equation (1), then this would correspond to $\gamma = 0$ in the more complete formulation of Reference 8. In such a case, Equation (39) would no longer have any meaning because the pressure p would not appear.

A more complete analytical development and discussion of the axisymmetric elastic buckling problem is given by Short and Pulos in Reference 20.

AXISYMMETRIC INELASTIC BUCKLING BETWEEN RING FRAMES

The collapse criteria which were developed earlier and were identified by the pressures p_{c1}, p_{c3}, p_{c5}, p_{c6}, p_{c7}, and p_{c8} are, strictly speaking, valid only for an elastic-perfectly plastic material. As it has already been stated, the axisymmetric collapse mode is in reality associated with the phenomenon of buckling at a reduced modulus, so that the strain-hardening characteristics as reflected by the secant (E_s) and tangent (E_t) modulii must be considered.

On the basis of the deformation theory of plasticity, Gerard[21] developed a general set of differential equations of equilibrium for cylindrical shells in which the coefficients reflect the plasticity or state-of-stress effects. Lunchick, specializes these equations for the case of short-length cylindrical shells subjected to hydrostatic pressure.[22] In

this way, he derives an expression for the inelastic buckling of ring-stiffened cylindrical pressure hulls.

The basic differential equation used by Lunchick in his analysis is

$$A_1 \frac{d^4w'}{dx^4} + 6(1-\nu^2) \frac{pR}{E_s h^3} \frac{d^2w'}{dx^2} + \frac{12}{R^2h^2} \left[A_2 - \nu \frac{A_{12}^2}{A_1} \right] w' = 0 \tag{42}$$

where the plasticity coefficients A_1, A_2, A_{12} and the variable Poisson ratio ν are expressed in terms of the elastic modulus E, secant modulus E_s, tangent modulus E_t, elastic Poisson ratio ν_e, and the prebuckling stress ratio k by the following expressions:

$$A_1 = 1 - \frac{(1-E_t/E_s)}{4(1-\nu^2)K^2H} \left[(2-\nu) - (1-2\nu)k\right]^2 \tag{43}$$

$$A_2 = 1 - \frac{(1-E_t/E_s)}{4(1-\nu^2)K^2H} \left[(1-2\nu) - (2-\nu)k\right]^2 \tag{44}$$

$$A_{12} = 1 + \frac{(1-E_t/E_s)}{4\nu(1-\nu^2)K^2H} \left[(2-\nu) - (1-2\nu)k\right] \left[(1-2\nu) - (2-\nu)k\right] \tag{45}$$

$$\nu = \frac{1}{2} - \left(\frac{1}{2} - \nu_e\right) \frac{E_s}{E} \tag{46}$$

and in which

$$k = \sigma_{\Phi m} / \sigma_{Xm}$$

$$K^2 = 1 - k + k^2 \tag{47}$$

$$H = 1 + \frac{(1-E_t/E_s)}{4(1-\nu^2)K^2} \left\{ \left[(2-\nu) - (1-2\nu)k\right]^2 - 3(1-\nu^2) \right\}$$

By assuming a deflection function corresponding to simple supports at the ring frames, i.e.,

$$w'(x) = w'_m \sin \frac{m\pi x}{L} \qquad (48)$$

Lunchick was able to derive the following expression for the buckling pressure:

$$p_{cr} = \frac{2}{(1-\nu^2)} \left(A_2 - \nu^2 \frac{A_{12}^2}{A_1} \right) C^2 E_s \left(\frac{h}{R} \right)^2 + \frac{1}{6(1-\nu^2)} \frac{A_1}{C^2} E_s \left(\frac{h}{R} \right)^2 \qquad (49)$$

where

$$C^2 = \frac{L^2}{\pi^2 Rh} \qquad (50)$$

In order to carry out numerical caluclations to find p_{cr} from Equation (49), a graphical solution must be resorted to. First, for a given geometry, the membrane state of stress at midbay defined by σ_{Xm} and $\sigma_{\Phi m}$ is determined using Equations (15) and (16), respectively, with σ_{xbm} set to zero therein. These stresses are then used in the Huber-Hencky-Von Mises theory of failure to compute a stress intensity σ_i, i.e.,

$$\sigma_i = \left[\sigma_{Xm}^2 + \sigma_{\Phi m}^2 - \sigma_{Xm} \sigma_{\Phi m} \right]^{1/2} \qquad (51)$$

Next, the uniaxial stress-strain curve for the material which comprises the cylinder is entered with the value of stress given by Equation (51), and values of E_s and E_t are found for this stress level. A value of p_{cr} can then be computed from Equation (49), with the aid of all the other equations, (43) through (47), needed to first find A_1, A_2, A_{12}, ν , ... etc. By repeat-

ing this process of computation, a curve of p vs σ_i can be found using Equation (51), and a corresponding curve of p_{cr} vs σ_i using Equation (49). These two curves are then plotted using pressure as the ordinate scale and stress intensity as the abscissa scale; the intersection of two curves gives the desired inelastic buckling pressure. A detailed discussion of this procedure is given in Reference 23 together with some pertinent curves for certain coefficients to help facilitate numerical computations.

ASYMMETRIC BEHAVIOR OF A RING-STIFFENED CYLINDRICAL PRESSURE HULL

Generally speaking, the three basic modes of collapse for ring-stiffened cylindrical pressure hulls are phenomenologically related to and influenced by instability. Therefore, the real problem of collapse is one in which the stress state and the propensity for instability mutually interact. Since premature collapse can be precipitated by instability, it is necessary that efficient design be based on judicious proportionment of the geometry for a given material, so that instability is prevented from occurring until the material is stressed well into the inelastic range.

Rational methods of analysis are therefore needed to accurately predict the elastic buckling of stiffened cylinders as a starting point from which the more complex interaction problem termed "inelastic buckling" can be solved more readily. Previous authors have suggested that an approach to efficient design may be based on the use of arbitrary factors or margins between buckling and desired maximum strength. Although this appears to be an oversimplification of the problem and represents a

philosophy not adhered to at the Model Basin, the starting point still goes back to rational methods of analysis for the elastic-instability problem.

At the outset, it is important to state categorically that, contrary to the belief of others, elastic instability of stiffened cylindrical shells under hydrostatic pressure is not obscured by the "snap-through" mechanism. This is in contrast to the problem of cylinders under axial compression or torsion loading where, as has been pointed out by Thielemann in an excellent paper on the nonlinear theories of buckling of thin cylindrical shells,[24] even for the case of no imperfections, the effects of large deformations are important.

A detailed investigation of the "Durchschlag" problem by Kempner et al[25] for cylindrical shells of short length which are perfectly circular and initially stress free has provided theoretical confirmation of the adequacy of classical small-deflection theory for predicting elastic-instability pressures. Experimental studies conducted at the Model Basin have provided test data which substantiate these theoretical findings; see for example, Reference 26. A more detailed discussion of the elastic-instability problem for ring-stiffened cylindrical shells is given by Reynolds,[27] with some emphasis on large-deflection theory versus small-deflection theory for the hydrostatic pressure case. Another excellent discussion of the instability problem has been given by Fung and Sechler.[28]

For the reasons emphasized above, no detailed consideration of the large-deflection problem will be given in this presentation, so that our

attention will be focused mainly on the small-deflection theories.

ELASTIC ASYMMETRIC (LOBAR) SHELL BUCKLING BETWEEN RING FRAMES

The first attempt at a rigorous solution of the elastic lobar-buckling problem of a thin cylindrical shell on the basis of the theory of elasticity, and later on the basis of thin-shell theory, was made by Southwell.[29] Although he concentrated his efforts on the case of radial pressure loading only, his work paved the way for the theoretical developments by those who followed after him, notably Von Mises[30] and Tokugawa.[31] An excellent discussion of the most important theoretical formulas developed by these three authors is given in Reference 32.

For our purposes, it suffices to say that the first satisfactory solution to the elastic buckling problem of a thin cylindrical shell of finite length is attributed to Von Mises.[30] He assumed simple-support boundary conditions at the ends of the shell, thus enabling him to obtain an exact solution to the thin-shell equations in closed form. At a later date, Windenburg and Trilling[32] simplified the original theoretical results of Von Mises, and this led to the development of some convenient formulas for design purposes. Some details covering both of these accomplishments will now be reviewed.

Instead of using the original notation of Von Mises, recourse will be made to the terminology and form of solution given by Timoshenko, details of which can be found in Reference 33. In terms of the displacements u, v, and w, the three differential equations of equilibrium for an

element of a cylindrical shell subjected to the simultaneous action of axial compression (N_x) and uniform lateral pressure (q) are given by

$$R^2 \frac{\partial^2 u}{\partial x^2} + \frac{1+\nu}{2} R \frac{\partial^2 v}{\partial x \partial \theta} - \nu R \frac{\partial w}{\partial x} + R\phi_1 \left(\frac{\partial^2 v}{\partial x \partial \theta} - \frac{\partial w}{\partial x} \right) + \frac{1-\nu}{2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{1+\nu}{2} R \frac{\partial^2 u}{\partial x \partial \theta} + \frac{1-\nu}{2} R^2 \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial \theta^2} - \frac{\partial w}{\partial \theta} + \bar{\alpha} \left[\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^3 w}{\partial \theta^3} + R^2 \frac{\partial^3 w}{\partial x^2 \partial \theta} + \right.$$

$$\left. + R^2 (1-\nu) \frac{\partial^2 v}{\partial x^2} \right] - R^2 \phi_2 \frac{\partial^2 v}{\partial x^2} = 0 \qquad (52)$$

$$\nu R \frac{\partial u}{\partial x} + \frac{\partial v}{\partial \theta} - w - \bar{\alpha} \left[\frac{\partial^3 v}{\partial \theta^3} + (2-\nu) R^2 \frac{\partial^3 v}{\partial x^2 \partial \theta} + R^4 \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial \theta^4} + \right.$$

$$\left. + 2R^2 \frac{\partial^4 w}{\partial x^2 \partial \theta^2} \right] = \phi_1 \left(w + \frac{\partial^2 w}{\partial \theta^2} \right) + \phi_2 R^2 \frac{\partial^2 w}{\partial x^2}$$

where

$$\frac{qR}{Eh}(1-\nu^2) = \phi_1; \ \frac{N_x}{Eh}(1-\nu^2) = -\phi_2; \ \bar{\alpha} = \frac{h^2}{12R^2} \qquad (53)$$

for the case of uniform external hydrostatic pressure, $\phi_2 = \frac{1}{2}\phi_1$ and $N_x = \frac{pR}{2}$.

Equations (52) are linear and homogeneous so that a solution for the displacements u, v, and w in terms of sine and cosine functions is possible. Von Mises assumed the following buckling displacements which satisfy the conditions of simple support at the ends of the cylinder:

$$u(x,\theta) = A \sin n\theta \cos \frac{m\pi x}{L}$$

$$v(x,\theta) = B \cos n\theta \sin \frac{m\pi x}{L} \tag{54}$$

$$w(x,\theta) = C \sin n\theta \sin \frac{m\pi x}{L}$$

where n and m are real integers denoting the number of full waves around the circumference and the number of half waves in the longitudinal direction, respectively, which form when the cylinder buckles. The buckling shape, Equation (54), may be considered as being the first terms in a more general double trigonometric-series solution. This general type solution will be discussed later in this section, but for our present purposes, attention will be restricted to the solution of, Equation (54). Substituting these expressions in Equations (52), we obtain for A, B, and C three homogenous linear algebraic equations. For a solution other than the trivial one (A = B = C = 0), the determinant formed by the coefficients of A, B, and C in these three algebraic equations must vanish. Expanding this 3x3 determinant and after some simplifications where only the linear pressure terms are retained, the resulting equation for calculating the critical values of pressure can be put in the following form:

$$C_1 + C_2\bar{\alpha} = C_3\phi_1 + C_4\phi_2 \tag{55}$$

in which

$$C_1 = (1-\nu^2)\lambda^4$$

$$C_2 = (n^2+\lambda^2)^4 - 2\left[\nu\lambda^6+3\lambda^4n^2+(4-\nu)\lambda^2n^4+n^6\right] + 2(2-\nu)\lambda^2n^2 + n^4 \tag{56}$$

$$C_3 = n^2(n^2+\lambda^2)^2 - (n^4+3\lambda^2n^2)$$

$$C_4 = \lambda^2(n^2+\lambda^2)^2 + \lambda^2 n^2$$

$$\lambda = \frac{m\pi R}{L} \qquad (56)$$

$$L = L_f - b$$

Assuming that the shell is thin and keeping only the principal terms in Equation (55), Von Mises obtained the following simplified formula for the critical value of the pressure:

$$p_{cr} = \frac{Eh}{R}\left[\frac{1}{n^2+\frac{1}{2}(\pi R/L)^2}\right]\left\{\frac{(\pi R/L)^4}{\left[n^2+(\pi R/L)^2\right]^2} + \frac{(h/R)^2}{12(1-\nu^2)}\left[n^2+\left(\frac{\pi R}{L}\right)^2\right]^2\right\} \qquad (57)$$

It is important to point out that the above equation results as a consequence of the assumption that n is large, say on the order of 10, so that then $(n^2-1) \approx n^2$. This implies that Equation (57) is not accurate for very long shells which buckle into the elliptic shape, i.e., $n = 2$, or even for relatively long shells which may have a critical buckling mode corresponding to $n = 3$, 4, or even 5. For such cases, which are not usually encountered in ring-stiffened cylindrical pressure hulls, recourse must be made to the more exact formula developed by Von Mises; see Equation (6) in Reference 32. For the limiting case of a cylinder of infinite length, i.e., $L \to \infty$, which buckles in the oval shape as does a "free" ring under radial loading, the following simple formula of Bresse and Bryan (see Reference 32) is applicable:

$$p_{cr} = \frac{E}{4(1-\nu^2)}\left(\frac{h}{R}\right)^3 \qquad (58)$$

Furthermore, Equation (57) is based on the fact that $m = 1$ which implies

that the cylinder buckles into a half-sine wave in the longitudinal direction. It can be shown numerically that lower values of p_{cr} result for $m = 1$ than for $m > 1$. Physical intuition also leads us to accept this because the buckling shape corresponding to $m = 1$ is associated with lower energy states than those for $m > 1$, at least for the case of hydrostatic pressure loading; however, this may not be so for the case of axial loading.

An examination of Equation (57) reveals that the critical pressure is dependent on the value of n. This means that for a given geometry of shell and for a given material, calculations must be conducted for different values of n in order to find that value of n which minimizes the pressure. It is this minimum pressure one seeks. To facilitate this calculation process, Von Mises developed a set of curves; these can be found in either Reference 30 or Reference 32.

Another approach to this minimization is to do it analytically and thus find an expression for p_{cr} which is independent of the parameter n. Windenburg and Trilling did exactly this in Reference 32, and the final convenient formula they arrived at is given by

$$p_{cr} = \frac{2.42E}{(1-\nu^2)^{3/4}} \left[\frac{(h/2R)^{5/2}}{\frac{L}{2R} - 0.45\left(\frac{h}{2R}\right)^{1/2}} \right] \qquad (59)$$

Calculations carried out for a range of L/2R from 1/8 to 2 and a range of h/2R from 0.002 to 0.007, for assumed values of $E = 30 \times 10^6$ psi and

$\nu = 0.3$, have shown that the maximum difference between predictions using Equations (59), (57), and the more exact formula, Equation (6), given in Reference 32 was about 3.5 percent.

The next major contribution to the panel buckling problem of ring-stiffened cylinders was that of Von Sanden and Tölke. By the use of trigonometric series,[33] they outlined a general solution to the same differential equations used by Von Mises. This approach also permits a closer approximation of the true prebuckling deformations given by the theory of either Reference 6 or Reference 8. However, Von Sanden and Tölke did not attempt to work out the mathematical details of the general solution, but they did develop a solution which was one step better than that of Von Mises. They assumed a two-term trigonometric approximation for the variability of the prebuckling circumferential stress with the axial coordinate, and with this they showed that the "simple-support functions" used by Von Mises for the buckling deformations permit satisfaction of the differential equations.

It is of interest to us here to give the final formula developed by Von Sanden and Tölke for purposes of comparison with that of Von Mises. In the notation adopted for this presentation, it can be shown to be as follows:

$$p_{cr} = \frac{Eh}{R}\left[\frac{1}{n^2\left(\frac{3}{4}\delta_m + \frac{1}{4}\delta_o\right) + \frac{1}{2}\left(\frac{\pi R}{L}\right)^2}\right]\left\{\frac{(\pi R/L)^4}{\left[n^2 + \left(\pi R/L\right)^2\right]^2} + \frac{(h/R)^2}{12(1-\nu^2)}\left[n^2 + \left(\frac{\pi R}{L}\right)^2\right]^2\right\} \qquad (60)$$

Equations (60) and (57) are of identical form; the only difference between the two is the factor

$$\left(\frac{3}{4}\delta_m + \frac{1}{4}\delta_o\right)$$

which multiplies n^2 in the denominator of Equation (60). The quantities δ_m and δ_o appearing in Equation (60) can be determined from the stress expressions given in an earlier section. It can be shown that using Equations (16) and (18), respectively,

$$\begin{aligned} \sigma_m &= 1 - \left(1 - \frac{\sigma_{\phi Mf}}{\sigma_u}\right) F_2 \\ \sigma_o &= 1 - \left(1 - \frac{\sigma_{\phi Mf}}{\sigma_u}\right) \end{aligned} \tag{61}$$

Of particular interest is the fact that when $\sigma_{\phi Mf} = \sigma_u$, then $\sigma_m = \sigma_o = 1$, and this reduces Equation (60) exactly to (57); the assumption that the prebuckling stress

$$\sigma_{\phi Mf} = \sigma_u \equiv -\frac{pR}{h}$$

was used by Von Mises. For closely framed cylinders where the stiffening effect is appreciable, Equation (60) may predict elastic buckling pressures on the order of as much as 50 percent higher than those of Equation (57). This suggests that the predictions from Equation (60) be used in any comparison between theory and experiment but that some conservatism in design of hulls can be introduced by the use of Equation (57).

A major contribution to the solution of shell-buckling problems was made by the group at the Polytechnic Institute of Brooklyn working under

the general direction of N. J. Hoff. In particular, Salerno, Levine, Pulos, and, later, Shaw and Bodner, developed formulations for elastic lobar (panel) buckling of ring-stiffened cylinders by application of the principle of minimum potential energy (Rayleigh-Ritz method). The approach was quite distinct from that used by the earlier investigators who attempted solutions of the differential equations. The energy method involves the assumption of buckling functions which satisfy some chosen ideal boundary conditions, and these are then used to satisfy the condition of minimum energy which implies satisfaction of the differential equations in this sense. It is of interest here to outline some of the more important results with regard to the application of this very potent method in the solution of shell-buckling problems of interest to naval architects.

First, expressions for the elastic strain energy in the shell and also in the ring frames are written in terms of the displacement components of a point in the middle surface of the shell. Then, expressions for the work done by the external pressure forces acting on the cylinder are also written in terms of these displacement components. Various displacement configurations for the buckled shell are then introduced to approximate the actual case. After the total potential is expressed in terms of these displacements containing arbitrary mode-shape parameters, the energy is then minimized with respect to these parameters and this process leads to a set of linear homogeneous algebraic equations. In order that a non-trivial solution to this system of equations exists, it is necessary that

the determinant of the coefficients vanish. This condition leads to a determinantal equation for finding the critical pressure at which elastic buckling of the cylindrical shell will occur. Some of the more important details will be illustrated later in connection with the elastic general-instability problem, the formulation of which follows along similar lines. However, the general energy approach to solving shell buckling problems is discussed in great detail by Timoshenko in Reference 9.

By using this energy method, Salerno and Levine[34] derived the Von Mises solution as a starting point. They also developed solutions for the cylindrical shell having clamped edges, and later attempted to include the bending and torsional restraints afforded the cylindrical shell by ring frames possessing finite elastic properties.[35] However, certain assumptions in their work led to some inconsistency in final results, so that a number of investigators after them used their basic formulation to get improved solutions. Shaw, Bodner, and Berks[36] have reviewed the whole problem of the energy approach to the panel-instability failure of reinforced cylindrical shells in an attempt to clarify some of the questions which arose in regard to the analytical work of Salerno and Levine.

The most recent and most useful solution derived by application of the Rayleigh-Ritz energy method is that by Reynolds.[27] He developed a solution in which the influence of the elastic ring frames on both the pre-buckling and buckling deformations was included. The use of a several-

term Fourier series, the convergence of which is dependent upon the shell-flexibility parameter θ or β, to express the deformations permitted a closer approximation to the true state of stress prior to buckling. By including the torsional as well as the bending energies of the ring frames, Reynolds was able to consider all degrees of elastic support afforded the shell by the ring frames, ranging between and including the two extreme cases of simple supports and clamped conditions.

The energy integrals and some of the intermediate mathematical operations used by Reynolds are rather lengthy and cumbersome so that these details will not be considered here. However, it is of interest to give the final equation from which the critical elastic panel-buckling pressures can be computed. Thus, for the most general case of interest to the designer, that of ring frames possessing finite bending and torsional stiffnesses, Reynolds derived the following buckling criterion:

$$\left[1 + \sum_i \frac{H_i U_i}{D_i}\right]\left[1 + \sum_i \frac{G_i \lambda_i}{D_i}\right] - \sum_i \frac{G_i U_i}{D_i} \sum_i \frac{H_i \lambda_i}{D_i} = 0 \tag{62}$$

where $i = 1, 3, 5, \ldots$ and, N is the summation index specifying the number of terms to be taken in the Fourier expansions to get varying degrees of numerical convergence. The other quantities appearing in Equation (62) are defined as follows:

$$H_i = \frac{2I_{zo} n^2}{R^2 L_f h\left(1+\frac{e}{R}\right)^3}\left\{\lambda_i\left[1-\frac{e}{R}(n^2-1)\right] + n^2 U_i\right\} + \frac{pR}{Eh}(2-\nu)\,\frac{\eta n^2}{\left(1+\frac{e}{R}\right)}\left[U_i - \frac{e}{R}\lambda_i\right] \tag{63}$$

$$U_i = -\frac{\lambda_i(n^2 - \nu\lambda_i^2)}{(n^2 + \lambda_i^2)^2} \tag{64}$$

$$D_i = \phi_i^2 + \frac{\lambda_i^4 (h/R)^2}{12(1-\nu^2)\phi_i^2} + \frac{pR}{2Eh}\left[\lambda_i^2 + n^2\left\{2-(2-\nu)\,\eta\left[1-a_i(1+8\phi_i^2)\right]\right\}\right] \tag{65}$$

$$G_i = \frac{2I_{zo}}{R^2 L_f h\left(1+\frac{e}{R}\right)}\left[1-\frac{e}{R}(n^2-1)\right]\left\{\lambda_i\left[1-\frac{e}{R}(n^2-1)\right] + n^2 U_i\right\} + \frac{Kn^2\lambda_i}{(1+\nu)R^2 L_f h\left(1+\frac{e}{R}\right)^3} +$$

$$+ \frac{pR}{Eh}(2-\nu)\frac{\eta}{(1+\frac{e}{R})}\left\{\lambda_i\left[\left(\frac{I_{xo}+A_f e^2}{R^2 A_f}\right)n^2 - \frac{e}{R}\left(1+\frac{e}{R}\right)\right] - n^2\frac{e}{R}U_i\right\} \tag{66}$$

and in which

$$\lambda_i = \frac{i\pi R}{L_f}\,;\ \phi_i = \frac{\lambda_i^2}{n^2+\lambda_i^2} \tag{67}$$

$$\eta = \left[\frac{L_f h}{A_f}\left(1+\frac{e}{R}\right) + \sum_{-\infty}^{\infty} a_m\right]^{-1} \tag{68}$$

$$a_m = \left\{\left[1+4\left(\frac{m\pi}{\beta}\right)^4\right]\left(1-p/p_m\right)\right\}^{-1} \tag{69}$$

$$p_m = \frac{2E(h/R)^2}{\sqrt{3(1-\nu^2)}}\left[\left(\frac{\beta}{2m\pi}\right)^2 + \left(\frac{m\pi}{\beta}\right)^2\right] \tag{70}$$

$$\beta = \sqrt[4]{3(1-\nu^2)}\ \ \frac{L_f}{\sqrt{Rh}} \tag{71}$$

The geometric quantities R, h, L_f, A_f, ... etc. are defined at the beginning of this report. The pressure p_m denotes the elastic buckling load for a cylinder which buckles axisymmetrically in a shape defined by

$$w(x) \sim \cos \frac{m\pi x}{L_f}.$$

Furthermore, Reynolds uses the notation that internal pressure is positive and external pressure is negative; therefore, for the particular case of external hydrostatic pressure loading which is of interest to us, it is necessary to substitute negative numbers for p wherever it appears in Equations (62) through (71). The multiplying factor $(1-p/p_m)$ appearing in Equation (69) reflects the "beam-column effect" introduced by Salerno and Pulos into the basic axisymmetric stress formulation.

It becomes obvious upon examination of the buckling equation (62) that it is transcendental in the pressure. Reynolds[27] suggests a graphical solution by which the left-hand side of Equation (62) is plotted against pressure. Such a plot will have $\frac{N+1}{2}$ zero - intercepts, one for each root of Equation (62), and an equal number of asymptotes corresponding to the vanishing of each of the denominators D_i. The first asymptote which results as a consequence of the denominator D_1 of Equation (62) vanishing corresponds to the buckling pressure for a single-wave simple support buckling configuration. The case $D_3 = 0$ corresponds to the buckling pressure for a three-wave simple support buckling configuration; and so on for $D_5 = 0$, etc. The first "non-singular root" of Equation (62) which

occurs between the first two asymptotes corresponding to $D_1 = 0$ and $D_3 = 0$ is the elastic buckling pressure for a stiffened cylinder with elastically restrained edges which buckles in a single-wave configuration. A more complete discussion of all the possible solutions of Equation (62) together with their physical interpretation is given in Reference 27.

SEMI-EMPIRICAL FORMULA FOR INELASTIC BUCKLING BETWEEN RING FRAMES

Before we get into the rigorous formulation of the inelastic panel-buckling problem, it is instructive to consider the derivation of a semi-empirical formula for cylindrical shells which follows along the same general development as for columns. The assumptions and underlying concepts are not new, but the order and emphasis given them should help to clarify a number of questions which have arisen in regard to this formula. The basic approach has been given by Trilling and Windenburg in an obscure reference in which they review the development of column formulas for inelastic buckling and suggest the extension of the same concepts for inelastic buckling of ring-stiffened cylindrical shells under hydrostatic pressure.

We start out first with a consideration of the column-buckling formulas. The well-known Euler formula for the buckling of columns in the elastic range is

$$P_E = \frac{\pi^2 EI}{\ell^2} \tag{72}$$

and since

$$\sigma_{cr_E} = \frac{P_E}{A},$$

the above equation leads to the following expression for the critical buckling stress:

$$\sigma_{cr_E} = \pi^2 E \rho^2 \qquad (73)$$

where

$$\rho = \frac{1}{\ell}\sqrt{\frac{I}{A}}.$$

When the nondimensional variables

$$S_{cr} = \frac{\sigma_{cr}}{\sigma_y} \; ; \; \lambda_E = \frac{1}{\pi\rho}\sqrt{\frac{\sigma_y}{E}} \qquad (74)$$

are introduced, the Euler formula can be written simply as

$$S_{cr_E} = \frac{1}{\lambda_E^2} \qquad (75)$$

The Euler formula, Equation (73), is a special case of the more general Considére-Engesser formula which may be written as

$$\sigma_{cr} = \pi^2 \overline{E} \rho^2 \qquad (76)$$

where $\overline{E}$ is known as the reduced modulus and is a function not only of the slope of the stress-strain diagram but, strictly speaking, also of the shape of the column cross section. However, it has been shown from rather extensive tests that the effect of the column cross section on $\overline{E}$ does not

vary greatly for sections ordinarily used which have an axis of symmetry perpendicular to the plane of bending.

Setting

$$\tau = \overline{E}/E \tag{77}$$

Equation (75) can be modified so that a general column-buckling formula applicable in both the elastic and inelastic ranges can be written, i.e.,

$$S_{cr} = \frac{\tau}{\lambda_E^2} \tag{78}$$

Thus, to determine the critical buckling stress from Equation (78), it is necessary to know τ, which, in turn, requires a knowledge of the shape of the stress-strain curve. We will now consider a method of determining τ.

Osgood[37] has developed the following general column-buckling formula:

$$S_{cr} = 1 - \frac{2}{j}\left(\frac{j}{j+2}\right)^{1+j/2} \lambda_E^j \tag{79}$$

Different values of the parameter j give rise to parabolas of different character. The case $j = 2$ represents a column formula suitable for ductile materials like mild steel. Substituting $j = 2$ in Equation (79) leads to the well-known Johnson parabola, i.e.,

$$S_{cr} = 1 - \frac{1}{4}\lambda_E^2 \tag{80}$$

Since the Johnson parabola is an empirical curve which "fits" well the experimental data in the inelastic-buckling range for columns of mild steel, it affords a means of finding τ analytically without knowing the actual stress-strain curve. Solving Equations (78) and (80) yields the

following expression for τ as a function of the critical stress:

$$\tau = 4S_{cr}(1-S_{cr}) \tag{81}$$

Equation (81) constitutes one of the important relations to be used in the derivation which follows.

In the preceding it has been shown that, in general, the critical buckling stress for columns is given by

$$S_{cr} = \tau S_{cr_E} \tag{82}$$

In the elastic case, $\tau = 1$ and Equations (78) and (82) reduce to the well-known results of Euler.

At this point of the derivation, a basic empiricism is introduced. In the case of plate structures, a slight modification of the form of Equation (82) in which τ is replaced by $\sqrt{\tau}$ is in better agreement with experiment so that

$$S_{cr} = \sqrt{\tau} S_{cr_E} \tag{83}$$

Trilling and Windenburg offer what appears to be a plausible explanation for the reduction in the critical buckling stress implied by Equation (83) for plate structures in the effect of lateral restraint offered by the second dimension. They go on further to say that "this substitution of $\sqrt{\tau}$ for τ in the case of tubes under end-loading is supported theoretically by Geckeler." Although these earlier investigations had not been extended to

the case of tubes under both radial and end-loading, recent experimental results lend some further validity to the use of Equation (83) for cylindrical shell structures subjected to hydrostatic pressure.

Elimination of τ from Equations (81) and (83) leads to the following expression for the critical stress in the inelastic buckling range for plate and shell structures:

$$S_{cr} = \frac{1}{1 + \frac{1}{4} S^2_{cr_E}} \tag{84}$$

In particular, Equation (84) is of interest in deriving an empirical formula for inelastic buckling of cylindrical shells.

In analogy to the Euler formula for elastic buckling of columns, we have the Von Mises formula, i.e., Equation (57), or the Von Sanden and Tölke formula, i.e., Equation (60), or even the Reynolds solution, i.e., Equation (62), for elastic panel buckling of ring-stiffened cylinders under hydrostatic pressure. At this point, the form of the equation for the elastic buckling pressure p_E is not as important as the assumption that the "average stress" for buckling in the elastic range is given by

$$\sigma_{cr_E} = \frac{p_E R}{h} \overline{B} \tag{85}$$

where $\overline{B}$ is a factor which can be determined from some appropriate formula resulting from the general expression for circumferential stress at midbay, i.e., Equation (16). The possibilities for $\overline{B}$, which reflects the reduction in circumferential stress at midbay due to the presence of the

stiffening rings over that in an unstiffened cylinder, are given by

$$\overline{B} = 1 - a\left[F_2 - \nu F_4 \sqrt{\frac{0.91}{1-\nu^2}}\right] \tag{86}$$

$$\overline{B} = 1 - aF_2 \tag{87}$$

where a has already been defined in connection with Equation (31). We see that Equation (86) is a consequence of the outer-fiber location, and Equation (87) is a consequence of the mid-fiber location. Another possibility results from Equation (29), i.e.,

$$\overline{B} = \frac{1}{1 + A_f / L_f h} \tag{88}$$

The choice is left to the discretion of the designer.

With Equation (85), the nondimensional stress ratio S_{cr_E} appearing in Equation (84) becomes:

$$S_{cr_E} \equiv \frac{\sigma_{cr_E}}{\sigma_y} = \frac{p_E}{\sigma_y h/R} \overline{B} \tag{89}$$

Substituting Equation (89) into Equation (84) leads to

$$S_{cr} = \frac{1}{1 + \left(\frac{\sigma_y h/R}{2p_E}\right)^2 \overline{B}^{-2}} \tag{90}$$

Equation (90) constitutes a more general relation than that developed by Trilling and Windenburg, who, in essence, assumed $\overline{B} = 1$. They assumed that the stress for elastic buckling to be that given by the simple hoop formula $\sigma = \frac{pR}{h}$; this neglects the stiffening action of the ring frames.

There still remains to formulate an expression for the nondimensional quantity S_{cr} which, as defined by Equation (74), is the ratio of the average stress at buckling to the yield stress. If we assume that the same general expression (Equation (85)) is valid in the inelastic range as well as in the elastic range, then the "average stress" for buckling in the inelastic range is

$$\sigma_{cr_I} = \frac{p_I R}{h} \overline{B} \tag{91}$$

so that then

$$S_{cr_I} \equiv \frac{\sigma_{cr_I}}{\sigma_y} = \frac{p_I}{\sigma_y h/R} \overline{B} \tag{92}$$

Substituting Equation (92) into the left-hand side of Equation (90) and solving for p_I leads to

$$p_I = \frac{(\sigma_y h/R) \overline{B}^{-1}}{1 + \left(\frac{\sigma_y h/R}{2p_E}\right)^2 \overline{B}^{-2}} \tag{93}$$

Equation (93) represents a more general semi-empirical formula for inelastic buckling of ring-stiffened cylindrical shells than that developed by Trilling and Windenburg. The general form of Equation (93) is similar to analogous formulas given by Timoshenko in Reference 9 for the inelastic buckling of straight columns and circular rings.

An important point to be made is that the same value of $\overline{B}$ must be used in both the numerator and denominator terms of Equation (93),

regardless of whether $\overline{B}$ is determined from Equation (86), (87), or (88). The significance of this will now be demonstrated.

Let us assume that the two values of $\overline{B}$ are different in Equation (93) and designate them $\overline{B}_N$ and $\overline{B}_D$ for the numerator and denominator terms, respectively. In the elastic buckling range, Equation (93) should reduce to the pressure p_E, i.e.,

$$p_I \equiv p_E = \frac{(\sigma_y h/R)\,\overline{B}_N^{-1}}{1+\left(\dfrac{\sigma_y h/R}{2p_E}\right)^2 \overline{B}_D^{-2}} \tag{94}$$

This leads to a quadratic equation in p_E which, when solved, yields

$$p_E = \frac{1}{2}\left(\frac{\sigma_y h}{R}\right)\left\{\overline{B}_N^{-1} \pm \sqrt{\overline{B}_N^{-2} - \overline{B}_D^{-2}}\right\} \tag{95}$$

where the lower of the double signs is to be considered. Now, taking the partial derivative of Equation (93), with the appropriate N and D subscripts on $\overline{B}$, with respect to p_E leads to the following expression for the slope of the p_I versus p_E curve:

$$\frac{\partial p_I}{\partial p_E} = \frac{\dfrac{1}{2}\left(\dfrac{\sigma_y h}{R}\right)^3 \overline{B}_N^{-1}\,\overline{B}_D^{-2}\,p_E}{\left[p_E^2 + \dfrac{1}{4}\left(\dfrac{\sigma_y h}{R}\right)^2 \overline{B}_D^{-2}\right]^2} \tag{96}$$

Substituting the value of p_E given by Equation (95), with the minus sign in front of the radical, into Equation (96) leads to the following:

$$\frac{\partial p_I}{\partial p_E} = \frac{(\overline{B}_N/\overline{B}_D)^2}{1-\sqrt{1-(\overline{B}_N/\overline{B}_D)^2}} \tag{97}$$

From Equation (97) it is seen that the slope of the p_I versus p_E curve at the point $p_I = p_E$ is equal to unity, as it should, only if $\overline{B}_N = \overline{B}_D$. Then, and only then, does the curve defined by Equation (93) become tangent to the p_E curve in the elastic buckling range. Thus, in the one extreme which corresponds to the elastic buckling range, Equation (93) for the inelastic buckling pressure p_I reduces to the elastic buckling pressure p_E, as it should. In the other extreme which corresponds to the axisymmetric collapse range, which conversely implies that $p_E \rightarrow \infty$, we see that Equation (93) reduces to $(\sigma_y h/R)\overline{B}^{-1}$, which is a hoop-stress relation at midbay between adjacent ring frames. As was indicated before, what to use for $\overline{B}$ is a question which has still not been completely resolved in the mind of the author. The question that needs to be answered is: "What is an appropriate hoop-stress criterion, if any exists, which adequately predicts axisymmetric collapse precipitated by yielding for the broad range of geometry which is of interest?" Some work to clarify this point is presently underway at the Model Basin; one possible approach toward resolving this question has been suggested by Pulos and Hom[38] in which empirical curves have been fitted through experimental data from many structural model tests and the trends toward asympotic values predicted by the various collapse criteria noted.

Equation (93) may appear to be somewhat nonrigorous to the more theoretically inclined stress analysts. A truly rigorous approach to solving the inelastic panel-buckling problem of ring-stiffened cylindrical

pressure hulls should start with an attempt to integrate the three partial differential equations for the shell deformations including elasto-plastic effects. This is the topic of discussion in the next section.

INELASTIC BUCKLING BETWEEN RING FRAMES

In recent years, advances in plasticity theory have made it possible to approach shell buckling problems more rigorously. Investigations by Biljaard,[39] Ilyushin,[40] Stowell,[41] and Gerard,[21] among others, have contributed greatly to the development of theory for the inelastic buckling of plates and shells; this school of thought made use of the deformation theory of plasticity.

Using the general set of differential equations for a fully plastic cylinder derived by Gerard in Reference 21, Reynolds[42] has developed a solution for the inelastic buckling of ring-stiffened cylinders, valid for both cases in which the shell material exhibits elastic-perfectly plastic and elastic strain-hardening characteristics. The buckling equations obtained by specializing the more general ones for the case of hydrostatic pressure loading are:

$$\frac{1}{2}\frac{E_t}{E_s}\left(\frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial s} + \frac{1}{R}\frac{\partial w}{\partial x}\right) + \frac{3}{4}\frac{\partial^2 u}{\partial x^2} + \frac{1}{4}\frac{\partial^2 u}{\partial s^2} + \frac{1}{4}\frac{\partial^2 v}{\partial x \partial s} = 0 \tag{98}$$

$$\frac{E_t}{E_s}\left(\frac{1}{2}\frac{\partial^2 u}{\partial x \partial s} + \frac{\partial^2 v}{\partial s^2} + \frac{1}{R}\frac{\partial w}{\partial s}\right) + \frac{1}{4}\frac{\partial^2 v}{\partial x^2} + \frac{1}{4}\frac{\partial^2 u}{\partial x \partial s} = 0 \tag{99}$$

$$\frac{4E_s h}{3R}\left(\frac{E_t}{E_s}\right)\left(\frac{1}{2}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial s}+\frac{w}{R}\right)+D\left[\frac{E_t}{E_s}\left(\frac{1}{4}\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w}{\partial x^2\partial s^2}+\frac{\partial^4 w}{\partial s^4}\right)+\right.$$

$$\left.+\frac{3}{4}\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w}{\partial x^2\partial s^2}\right]+N_x\frac{\partial^2 w}{\partial x^2}+N_s\frac{\partial^2 w}{\partial s^2}+p=0 \tag{100}$$

where x and s are, respectively, the axial and circumferential coordinates and N_x and N_s are forces per unit length in the axial and circumferential directions, respectively.

By following a procedure similar to that used by Donnell, Reynolds was able to combine Equations (98) through (100) into a single eighth-order equation in the radial displacement w only; this result is given as

$$D\left\{\frac{E_t}{E_s}\nabla^8 w+\left(1-\frac{E_t}{E_s}\right)\left[\nabla^4\left(\frac{3}{2}\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w}{\partial x^2\partial s^2}\right)-\frac{3}{4}\left(1-\frac{E_s}{E_t}\right)\left(\frac{3}{4}\frac{\partial^8 w}{\partial x^8}+\frac{\partial^8 w}{\partial x^6\partial s^2}\right)\right]\right\}+$$

$$+\frac{E_s h}{R^2}\frac{\partial^4 w}{\partial x^4}+N_x\left[\nabla^4\left(\frac{\partial^2 w}{\partial x^2}\right)-\frac{3}{4}\left(1-\frac{E_s}{E_t}\right)\frac{\partial^6 w}{\partial x^6}\right]+N_s\left[\nabla^4\left(\frac{\partial^2 w}{\partial s^2}\right)-\frac{3}{4}\left(1-\frac{E_s}{E_t}\right)\frac{\partial^6 w}{\partial x^4\partial s^2}\right]=0 \tag{101}$$

where ∇^4 indicates the operator $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial s^2}\right)^2$

Reynolds assumed the following buckling shape as a solution to Equation (101):

$$w(x,s) = A\sin ks\,\sin\lambda x \tag{102}$$

where $k = \frac{n}{R}$; $\lambda = \frac{m\pi}{L}$; m and n are integers which denote the number of half-waves in the longitudinal direction and the number of full waves in the circumferential direction, respectively. Equation (102) satisfies the conditions of simple support at the ends of the cylinder, i.e., that w and $\frac{\partial^2 w}{\partial x^2}$ vanish at the ends x = 0 and x = L. These conditions are not unreasonable for stiffened cylinders since it is likely that the effective rotational restraint will be limited by the formation of plastic regions arising from high bending stresses near the ring frames.

By substituting Equation (102) into the differential equation (101), Reynolds obtained a characteristic-value equation, which using the following notation

$$\begin{aligned}
\phi &= \frac{\lambda^2}{\lambda^2 + k^2} = \frac{1}{1 + \left(\frac{nL}{m\pi R}\right)^2} \\
f_p &= \frac{pR}{2N_s} = \frac{\sigma_x}{\sigma_s} \\
C &= \frac{E_s}{E_t} - 1
\end{aligned} \tag{103}$$

and rearranging terms to solve for the plastic buckling pressure p_p, leads to

$$p_p = \frac{2f_p D\lambda^2 \dfrac{E_t}{E_s}\left\{1 + C\phi\left[1 + \frac{\phi}{2} + \frac{3}{4}C\phi^2\left(1 - \frac{\phi}{4}\right)\right]\right\} + 2\,\dfrac{E_s h f_p \phi^4}{R^2\lambda^2}}{R\phi\left[1 - \phi\left(1 - f_p\right)\right]\left[1 + \frac{3}{4}C\phi^2\right]} \tag{104}$$

Equation (104) expresses the plastic buckling pressure p_p in terms of m and n. Reynolds minimized this expression with respect to ϕ; that is, he set $\partial p_p/\partial\phi = 0$, and by devising a graphical scheme, he was able to obtain the following convenient formula for the plastic buckling pressure:

$$p_p = \frac{8\pi^2 E_t f_p}{9\phi}\left(\frac{h}{R}\right)^2\left(\frac{\sqrt{Rh}}{L}\right)^2\left[\frac{1-\frac{3}{4}\phi(1-E_s/E_t)}{3-2\phi(1-f_p)}\right] \tag{105}$$

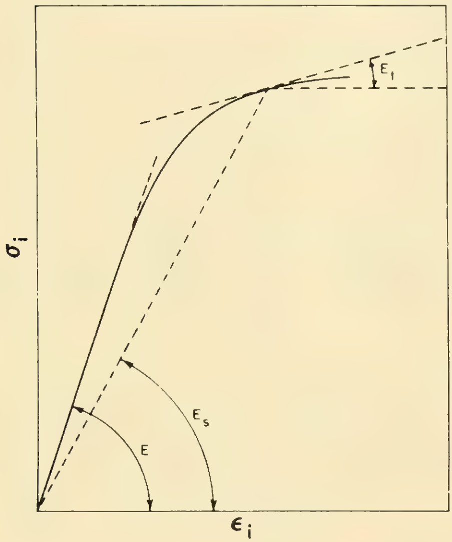


Figure 13 - Typical Stress-Strain Diagram Showing the Various Modulii of Interest

Before Equation (105) can be used, the secant and tangent modulii, which are defined on Figure 13, and are given, respectively, by

$$E_s = \frac{\sigma_i}{\varepsilon_i}\,;\; E_t = \frac{d\sigma_i}{d\varepsilon_i} \tag{106}$$

must be related to the applied pressure. The procedure is the same as that outlined for finding numerical solutions of Equation (49) to determine the axisymmetric inelastic buckling pressure. A detailed discussion of the procedure is given in Reference 42. It is important to point out that these formulations for inelastic buckling of stiffened cylinders can be applied to either case where the material is of the strain-hardening type or of the ideally plastic type; this corresponds to curvilinear and plateau stress-strain curves, respectively, in the plastic range.

The final equations used by Reynolds to relate stress-intensity σ_i

where

$$\sigma_i = \left(\sigma_X^2 + \sigma_S^2 - \sigma_X \sigma_S\right)^{1/2} \tag{107}$$

to the state of stress in a ring-stiffened cylinder, using the linear theory of Von Sanden and Günther[6] instead of that due to Salerno and Pulos,[8] are as follows:

$$\sigma_X = \frac{pR}{2h}$$

$$\sigma_S = \frac{pR}{h}\left[1 - \frac{\frac{3}{4}\left(A_f / L_f h\right)\left(\beta_p' - \frac{1}{2}\beta_p\right)}{\frac{1}{2}\beta_p\left(\frac{A_f + bh}{Lh}\right) + 1}\right] \tag{108}$$

where:

$$\beta_p = \theta_p\left(\frac{\sinh\theta_p + \sin\theta_p}{\cosh\theta_p - \cos\theta_p}\right)$$

$$\beta_p' = \frac{\theta_p}{2}\left(\frac{\sinh\frac{\theta_p}{2} + \sin\frac{\theta_p}{2}}{\cosh\frac{\theta_p}{2} - \cos\frac{\theta_p}{2}}\right) \tag{109}$$

$$\theta_p = (2.25)^{1/4}\ L/\sqrt{Rh}$$

Thus, using Equations (107) through (109), one can calculate a value of stress-intensity σ_i as a linear function of applied pressure p; this is plotted as a straight line on a p versus σ_i plot. Next, the uniaxial stress-strain curve of the material is entered with the value of σ_i given by Equation (107), and this determines E_s and E_t. These values are then used in conjunction with Equations (103) to determine the plastic buckling pressure p_p from Equation (105). In this fashion, a plot of p_p versus σ_i is obtained. The intersection of the two curves, p versus σ_i and p_p versus σ_i, gives the desired value of plastic buckling load. This is shown in Figure 14 for the two general classes of material mentioned before.

Equations (105) through (109) define the buckling pressure for the fully plastic case corresponding to $\nu = \frac{1}{2}$. By employing an empirical correction factor wherein Poisson's ratio is regarded as a variable, one can arrive at an expression which defines the buckling pressure in the inelastic range. Gerard and Wildhorn[43] have found that ν can be accurately expressed as a function of E_s in the inelastic region by the equation

$$\nu = \frac{1}{2} - \frac{E_s}{E}\left(\frac{1}{2} - \nu_e\right) \tag{110}$$

which reduces to 1/2 when E_s/E is zero and to the elastic value ν_e when E_s/E is unity. For the general inelastic case where ν is a variable defined by Equation (110), Reynolds gives the following formula for buckling pressure:

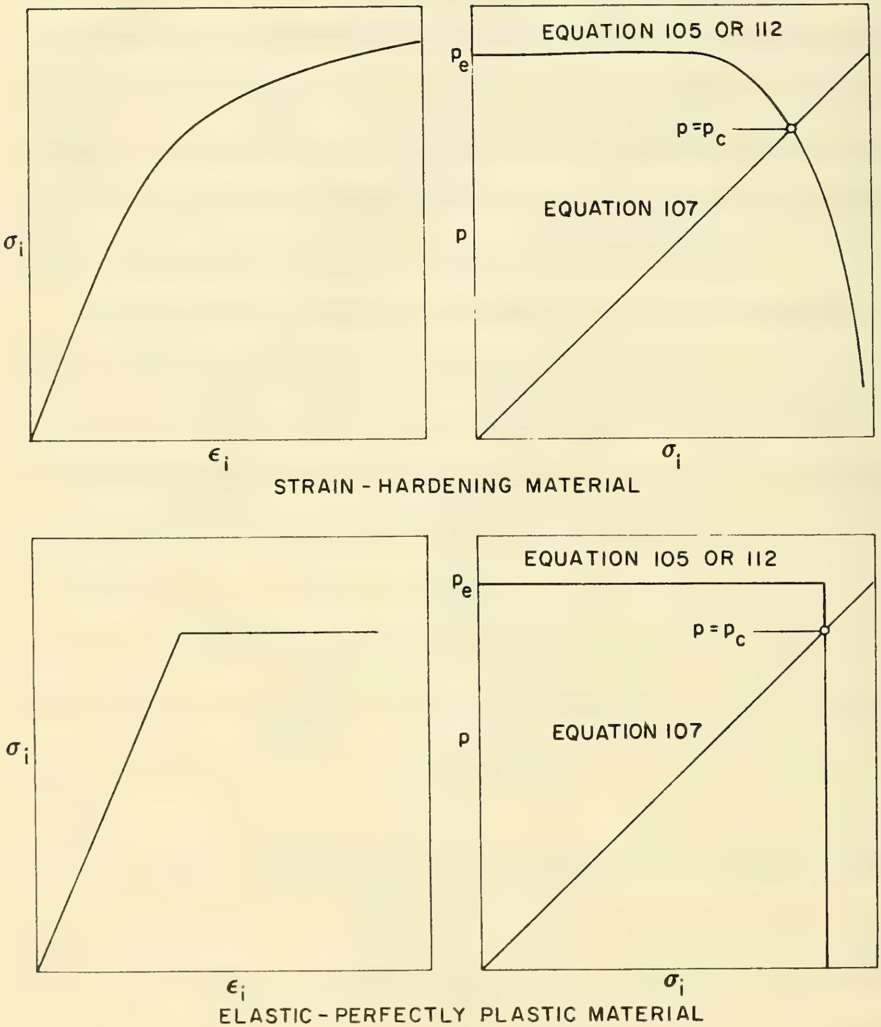


Figure 14 - Graphical Determination of Inelastic Buckling Pressure for Two General Classes of Material

$$p_c = \frac{2\pi^2 E_t f}{3\phi(1-\nu^2)} \left(\frac{h}{R}\right)^2 \left(\frac{\sqrt{Rh}}{L}\right)^2 \left[\frac{1-\frac{3}{4}\phi(1-E_s/E_t)}{3-2\phi(1-f)}\right] \tag{111}$$

In determining the stress ratio $f = \frac{\sigma_X}{\sigma_S}$, the stress σ_X is still equal to $\frac{pR}{2h}$ since it comes about from simple equilibrium considerations, and the stress σ_S is again taken to be that midway between ring frames as given by the theory of Von Sanden and Günther,[6] but with ν a variable defined by Equation (110). It is found, however, that σ_S is practically insensitive to variations in ν and that it is sufficient to treat f as a constant which depends only on the geometry of the cylinder; thus ν_e can be used for determining the stress ratio f.

The final equations given by Reynolds in Reference 42 for determining the inelastic buckling pressure p_c are as follows:

$$p_c = p_e\left(\frac{1-\nu_e^2}{1-\nu^2}\right)\left[\frac{E_t}{E}\left(1 - \frac{3}{4}\phi\right) + \frac{3}{4}\phi\frac{E_s}{E}\right] \tag{112}$$

where

$$p_e = \frac{2\pi^2 E f_e}{3\phi(1-\nu_e^2)} \left(\frac{h}{R}\right)^2 \frac{(\sqrt{Rh}/L)^2}{3-2\phi(1-f_e)} \tag{113}$$

$$\phi = 1.23\frac{\sqrt{Rh}}{L} \tag{114}$$

$$f_e = \frac{1}{2}\left[1 - \frac{\left(1-\frac{\nu_e}{2}\right)\left(A_f/L_f h\right)\left(\beta_e' - \frac{1}{2}\beta_e\right)}{1 + \frac{1}{2}\beta_e\left(\frac{A_f + bh}{Lh}\right)}\right]^{-1} \tag{115}$$

and

$$\beta_e = \theta_e \left(\frac{\sinh\theta_e + \sin\theta_e}{\cosh\theta_e - \cos\theta_e} \right)$$

$$\beta_e' = \frac{\theta_e}{2} \left(\frac{\sinh\frac{\theta_e}{2} + \sin\frac{\theta_e}{2}}{\cosh\frac{\theta_e}{2} - \cos\frac{\theta_e}{2}} \right) \tag{116}$$

$$\theta_e = \sqrt[4]{3(1-\nu_e^2)}\ L/\sqrt{Rh}$$

The method of calculation has already been discussed in connection with Equation (105); the same basic approach applies to Equation (112).

It is of interest to mention that out of this general solution resulting from the use of Equation (101), Reynolds was also able to derive an elastic buckling solution by using ν_e in place of ν and setting $E_s = E_t = E$. This solution, similar to that of Von Sanden and Tölke, i.e., Equation (60), is given by Equation (113).

FAILURE CRITERION FOR COLLAPSE OF AN IMPERFECTLY CIRCULAR SHELL

Using classical small-deflection theory, Galletly and Bart[44] carried out a theoretical investigation of the effects of boundary conditions and initial out-of-roundness on the strength of cylinders subject to external hydrostatic pressure. The equations developed by these authors were applied to the case of initially out-of-round cylinders with clamped ends; these equations represented a slightly modified form of those previously derived by Bodner and Berks,[45] working at the Polytechnic Institute of

Brooklyn, for cylinders with simply supported ends.

The approach used by Galletly and Bart was similar to that of Bodner and Berks except that instead of attempting to find a solution directly to the Donnell-type shell equations, they employed Galerkin's method in conjunction with the Donnell equations modified to include the effects of eccentricity. One of the limitations of using Donnell's equations is that the number of circumferential lobes n should be fairly high, and thus the results will be somewhat in error for very long cylinders which buckle into two or three circumferential lobes, i.e., n = 2 and n = 3, respectively.

The initial out-of-roundness pattern assumed by Bodner and Berks was of the form

$$w_o(x, \theta) = e \sin n\theta \cos \frac{\pi x}{L} \tag{117}$$

(origin at midbay) while Galletly and Bart assumed

$$w_o(x,\theta) = \frac{e}{2} \sin n\theta \left[1 - \cos \frac{2\pi x}{L}\right] \tag{118}$$

(origin at one end of the cylinder). Thus, in both cases, the initial out-of-roundness was similar in form to one of the assumed buckling modes. The two solutions represent lower and upper bounds, respectively, for the effect of initial eccentricities on the collapse pressures of elastically supported cylinders when the initial eccentricities have the same shape as one of the assumed buckling modes of the perfect cylinder.

For the case of uniform external hydrostatic pressure, Galletly and

Bart derived the following relevant partial differential equations in Reference 44, using the principle of minimum potential energy:

$$D\nabla^4 w + \frac{Eh}{R^2}\nabla^{-4} w,_{xxxx} + pR\left[\frac{1}{2}(w+w_o),_{xx} + \frac{1}{R^2}(w+w_o),_{\theta\theta}\right] - p = 0 \qquad (119)$$

$$\nabla^4 u = \frac{1}{R}\left[\nu w,_{xxx} - \frac{1}{R^2} w,_{x\theta\theta}\right] \qquad (120)$$

$$\nabla^4 v = \frac{1}{R^2}\left[(2+\nu)\, w,_{xx\theta} + \frac{1}{R^2} w,_{\theta\theta\theta}\right] \qquad (121)$$

$$\nabla^4 F = -\frac{E}{R} w,_{xx} \qquad (122)$$

where

$$\nabla^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2}\right)^2$$

$$\nabla^4 \nabla^{-4} f = f$$

F is the stress function of the total membrane stresses, and

w_o the initial out-of-roundness, is + inward toward the axis of the cylinder.

The comma after the dependent variables denotes partial differentations with respect to the independent variable which follows, so that

$$w,_{xxxx} \equiv \frac{\partial^4 w}{\partial x^4}\,;\; w,_{x\theta\theta} \equiv \frac{\partial^3 w}{\partial x\,\partial\theta^2}\,;\; \ldots \text{ etc.}$$

The assumed functions for the buckling displacements w and initial out-of-roundness w_o for clamped-end cylinders are, respectively,

$$\begin{aligned} w(x,\theta) &= B \sin n\theta \left[1 - \cos \frac{2\pi x}{L}\right] \\ w_o(x,\theta) &= \frac{e}{2} \sin n\theta \left[1 - \cos \frac{2\pi x}{L}\right] \end{aligned} \tag{123}$$

where e is the maximum amplitude of out-of-roundness and B is the buckling coefficient. If the expressions (123) happen to be an exact solution of the problem, they will satisfy the differential equation of equilibrium, Equation (119), exactly. However, as both w and w_o were chosen to satisfy the boundary conditions rather than the equilibrium equation, this, in general, will not be the case. The resulting expression will be a function of x and θ which we shall denote by Q. In such a case, Galerkin's method is used for determining the relations between the coefficients B and e; this leads to the following condition:

$$\int_0^{2\pi}\int_0^{L} Q \sin i\theta \left[1 - \cos \frac{2\pi x}{L}\right] R\,d\theta\,dx = 0\,;\; i = 1,2,3, \ldots \text{ etc.} \tag{124}$$

For $i \neq n$, Equation (124) will be found to be zero identically. For $i = n$ the following relation between B and e is obtained from Equation (124):

$$B = \frac{e}{2}\left(\frac{p}{p_{cr} - p}\right) \tag{125}$$

where p_{cr} is the buckling pressure of the perfect cylinder and is given by

$$p_{cr} = \frac{Eh}{R}\left(\frac{1}{3n^2 + \frac{\Delta^2}{2}}\right)\left\{\frac{\Delta^4}{(n^2+\Delta^2)^2} + \frac{(h/R)^2}{12(1-\nu^2)}(3n^4 + 2n^2\Delta^2 + \Delta^4)\right\} \tag{126}$$

and $\Delta = \frac{2\pi R}{L}$. The smallest value of the buckling pressure p_{cr} is found by minimizing Equation (126) with respect to n. A similar relation for the elastic buckling of a cylinder with clamped ends was derived by Nash[46] using an energy method. Bodner and Berks[45] derive a relation similar to Equation (125) for simply supported imperfect cylinders.

Combining Equations (123) and (124), we obtain the following expression for w:

$$w(x,\theta) = \frac{e}{2}\left(\frac{p}{p_{cr}-p}\right) \sin n\theta \left[1 - \cos\frac{2\pi x}{L}\right] \tag{127}$$

The bending moments in the shell can then be calculated from the well-known relations

$$\begin{aligned} M_x &= -D\left[w,_{xx} + \frac{\nu}{R^2} w,_{\theta\theta}\right] \\ M_\theta &= -D\left[\frac{1}{R^2} w,_{\theta\theta} + \nu w,_{xx}\right] \end{aligned} \tag{128}$$

The maximum bending stresses are then given by

$$\begin{aligned} \sigma_{bx} &= \pm \frac{6}{h^2}(M_x)_{max.} \\ \sigma_{b\theta} &= \pm \frac{6}{h^2}(M_\theta)_{max.} \end{aligned} \tag{129}$$

To obtain the total normal stresses, we have to add the membrane stresses

to Equations (129). To determine these latter, we substitute Equation (127) into Equation (122) and integrate for the stress function F, retaining only the periodic terms. The total membrane stresses are then given by

$$\begin{aligned} \sigma_{mx} &= -\frac{pR}{2h} + \frac{1}{R^2} F,_{\theta\theta} \\ \sigma_{m\theta} &= -\frac{pR}{h} + F,_{xx} \end{aligned} \tag{130}$$

The total normal stresses are obtained by adding algebraically Equations (129) and (130). The greatest normal stresses occur at midbay of the cylinder ($x = \frac{L}{2}$) and where $\sin n\theta = \pm 1$, which corresponds to the trough and crest points of the lobar pattern, respectively. At these points, the twisting moment $M_{x\theta}$ is zero and thus the normal stresses are principal stresses. The absolute maximum normal stresses occur at the outer shell wall for the trough points.

Having obtained the maximum principal stresses σ_X and σ_Θ in terms of the amplitude of out-of-roundness, the geometric parameters of the cylinder, and the applied pressure, the Hencky-Huber-Von Mises criterion of failure discussed earlier, see Equation (34), is employed to determine the pressure at which yielding initiates at the most critically stressed point. Thus, substitution of the maximum principal stresses σ_X and σ_Θ into

$$\sigma_y^2 = \sigma_X^2 + \sigma_\Theta^2 - \sigma_X \sigma_\Theta \tag{131}$$

gives an equation relating the initial out-of-roundness, the geometric parameters of the cylinder, the yield point σ_y of the material, and the

pressure p_y at which the shell begins to yield.

A general set of relations valid for both the simply supported and the clamped cylinders was given in Reference 44 as follows:

$$\frac{e}{h} = \left(1 - \frac{p_y}{p_{cr}}\right)\left\{\frac{-3 + \left[9 - 4(1 - \beta + \beta^2) H\right]^{1/2}}{4(1 - \beta + \beta^2) K}\right\} \times \begin{cases} 1 \text{ for simple supports} \\ 2 \text{ for clamped ends} \end{cases} \qquad (132)$$

where p_{cr} for the simply supported shell may be determined from the Von Mises Formula (57), and that for the clamped shell from Equation (126). Also for the simply supported shell,

$$\beta = \frac{\nu n^2 + \delta^2 + 2(1-\nu^2)\left(\frac{R}{h}\right)\frac{n^2\delta^2}{\left(n^2+\delta^2\right)^2}}{n^2 + \nu\delta^2 + 2(1-\nu^2)\left(\frac{R}{h}\right)\frac{\delta^4}{\left(n^2+\delta^2\right)^2}}$$

$$\frac{2p_{cr}(1-\nu^2)}{E}\left(\frac{R}{h}\right)^3 K = n^2 + \nu\delta^2 + 2(1-\nu^2)\left(\frac{R}{h}\right)\frac{\delta^4}{\left(n^2+\delta^2\right)^2} \qquad (133)$$

where $\delta = \frac{\pi R}{L}$. Whereas for the clamped shell,

$$\beta = \frac{\nu(2n^2-1) + \triangle^2 + 2(1-\nu^2)\left(\frac{R}{h}\right)\frac{n^2\triangle^2}{\left(n^2+\triangle^2\right)^2}}{2n^2 - 1 + \nu\triangle^2 + 2(1-\nu^2)\left(\frac{R}{h}\right)\frac{\triangle^4}{\left(n^2+\triangle^2\right)^2}}$$

$$\frac{2p_{cr}(1-\nu^2)}{E}\left(\frac{R}{h}\right)^3 K = 2n^2 - 1 + \nu\triangle^2 + 2(1-\nu^2)\left(\frac{R}{h}\right)\frac{\triangle^4}{\left(n^2+\triangle^2\right)^2} \qquad (134)$$

where $\triangle = \frac{2\pi R}{L}$. Also, for both cases

$$H = 3 - \left(\frac{2h\sigma_y}{Rp_y}\right)^2 \tag{135}$$

Galletly and Bart go into considerable discussion of various methods for determining initial out-of-roundness from shape profiles. They also carry out calculations for specific cylinders using Equations (132) through (135), and compare their results with experimental data obtained at the Model Basin. The reader is referred to Reference 44 for further details of the mathematical analysis and method of numerical computation. However, it is worth summarizing here that the best semi-empirical method for determining the "out-of-roundness" components from circularity profiles is that due to Holt; this technique essentially "picks out" the Fourier component corresponding to the buckling mode of the perfect cylinder. For a more exact determination of the contribution due to each component, a general harmonic analysis can be conducted on the most complicated circularity contour, and the method of superposition can then be used to determine the total "out-of-roundness" stress at any location.

ELASTIC GENERAL INSTABILITY OF SHELL AND RING FRAMES

Implicit in the formulations considered so far is the premise that the ring frames possess adequate bending and torsional rigidities, thus precluding local instability or failure of the stiffeners themselves. The influence of only the finite elastic properties of the ring frames on the structural behavior of the shell has been included in the theories discussed in the preceding sections. However, it is conceivable that if the ring

frames are "light," in the sense that they are not of sufficient cross section and/or of sufficient inertia, and if the overall length of the cylinder is large, say exceeding something on the order of two diameters or more, then another mode of collapse may intervene.

If this situation exists, then local frame failure could precipitate "bodily collapse" or general instability of the shell and ring frames. This possibility was first recognized by Tokugawa.[31] Just as in the case of shell lobar instability, this overall mode could possibly lead to premature collapse of a pressure hull unless the design is based on considerations governed by those geometric parameters which do not influence the axisymmetric and panel-instability collapse behaviors.

Tokugawa's[31] was the first rational attempt at developing some theory to predict the general-instability behavior. His approach was essentially based on a method which has come to be known in modern terminology as the "method of split rigidities." Much later, Bryant[47] working at the Naval Construction Research Establishment, Dunfermline, Scotland, arrived at practically the same end result but from a different point of view. This latter formulation was based on considerations of the elastic potential-energy of the shell and ring-frame system; however, the "true" interaction between shell and ring frames was approximated by Bryant.

For our purposes here, it suffices to give the final Tokugawa-Bryant formula. In the notation of this report, it takes the following form:

$$p_{cr} = \frac{Eh}{R}\left[\frac{\lambda^4}{\left(n^2 - 1 + \frac{\lambda^2}{2}\right)\left(n^2 + \lambda^2\right)^2}\right] + (n^2 - 1)\,\frac{EI_e}{R^3 L_f} \tag{136}$$

where $\lambda = \frac{\pi R}{L_b}$. It has become common practice at the Model Basin to compute I_e as the moment of inertia of the combined section of one ring frame plus an effective length L_e (see Equation (22)) of shell, instead of a full-frame-spacing L_f as was originally suggested by Bryant. This, in a sense, compensates for neglecting the true load interaction between shell and ring frames, (as was done by Bryant in his analysis.) It should be pointed out that Tokugawa's original equation was somewhat more complicated than that given above; however, calculations for a wide range of interest indicate that the additional terms included by Tokugawa were practically insignificant. As a matter of fact, it usually turns out that the second term of Equation (136) is dominant in the case of hull structures designed for shallow depths. It is usual to refer to the first term of Equation (136) as a shell term and to the second as a ring-frame term, in accordance with the "split-rigidities" concept.

Equation (136) as it stands does not permit discrimination between external and internal ring frames; the second term in Equation (136) is based on the assumption that the entire cross section of the ring frame is concentrated at the median surface of the shell plating. If one goes back to the basic formulation of the ring-buckling problem, it is rather easy to ascertain that a more correct ring-frame term for use in Equation (136) can be arrived at from

$$q_f \equiv p_f R_o L_e = (n^2 - 1) \frac{EI_e}{R_{cg}^2}$$

so that

$$p_f = (n^2 - 1) \frac{EI_e}{R_o R_{cg}^2 L_f} \qquad (136')$$

where R_o is the radius to the outstanding flange for an external ring-frame, but however, is defined as the radius to the contact surface with the shell plating for an internal ring frame; and R_{cg} is the radius to the centroid of the combined cross section made up of one frame and an "effective length" L_e of shell plating. The quantity L_e can be computed using Equation (22).

Just as in the case of Equation (57), it is necessary to minimize the critical buckling pressure p_{cr} of Equation (136) with respect to the number of circumferential lobes n. To facilitate calculations, Ball[48] has developed a graphical solution of Equation (136); further discussion of this will be given later in connection with some of the more complete formulations of the general-instability problem.

The next major theoretical development for this problem was that of the group at the Polytechnic Institute of Brooklyn, notably the work of Salerno and Levine.[49] Their method of solution was based on the principle of minimization of the potential energy as for the shell (lobar) buckling problem. The same general system of energy expressions was used in both cases; however, in the general-instability problem, the total energy

of the shell is integrated over the entire length L_b of the cylinder, and the ring energies are summed over the total number of ring frames. Bodner and Shaw[50] review the whole problem of the energy approach to both lobar (sometimes referred to as panel) and general-instability failure of reinforced cylindrical shells.

The work of Salerno and Levine constituted the basis for theoretical developments by investigators who followed them. The most acceptable solution of the elastic general-instability problem is that attributed to Kendrick[51] of the Naval Construction Research Establishment. Extensive confirmation of this theory has been reported by Reynolds and Blumenberg[52] at the Model Basin. It is of interest here to summarize the basic equations and integrals used by Kendrick in his formulation.

The total potential energy V_T of the elastic system, comprised of the cylindrical shell and the ring frames, is given by

$$V_T = U_e + U_b + \sum_{r=1}^{N} (F_e)_r + \sum_{r=1}^{N} (F_b)_r + W \qquad (137)$$

where

U_e and U_b are the extensional and bending strain energies of the shell, respectively

F_e and F_b are the extensional and bending strain energies of the ring frames, respectively

W is the total work done on the elastic system by the external loads due to the pressure, and

N is the total number of ring frames on the cylinder.

These quantities, as used in Reference 51, are defined by the following integrals in terms of the displacements u(x,θ), v(x,θ), and w(x,θ):

$$U_e = \frac{ERh}{2(1-\nu^2)} \int_0^{2\pi}\int_0^{L_b} \Big[u,_x^2 + (v,_\theta - w)^2/R^2 + 2\nu u,_x (v,_\theta - w)/R +$$

$$+ \frac{(1-\nu)}{2} (v,_x + u,_\theta/R)^2 \Big] d\theta dx + \frac{N_{oy}}{2R} \int_0^{2\pi}\int_0^{L_b} (u,_\theta^2 + v,_\theta^2 - 2v,_\theta w)\, d\theta dx +$$

$$+ \frac{RN_{ox}}{2} \int_0^{2\pi}\int_0^{L_b} (v,_x^2 + w,_x^2)\, d\theta dx \tag{138}$$

$$U_b = \frac{Eh^3}{24R(1-\nu^2)} \int_0^{2\pi}\int_0^{L_b} \Big[R^2 w,_{xx}^2 + (w,_{\theta\theta} + w)^2/R^2 + 2\nu w,_{xx}(w,_{\theta\theta} + w) +$$

$$+ 2(1-\nu)\left(w,_{x\theta} + \frac{1}{2} v,_x - \frac{1}{2R} u,_\theta\right)^2 \Big] d\theta dx \tag{139}$$

where the subscripts x and θ after a comma denote partial differentation with respect to the variable, i.e.,

$$v,_\theta \equiv \frac{\partial v}{\partial \theta}\,;\; w,_{x\theta} \equiv \frac{\partial^2 w}{\partial x \partial \theta}\,;\; \text{etc.}$$

The quantities N_{ox} and N_{oy} are the longitudinal and circumferential (hoop) thrusts per unit length, respectively, just prior to buckling. The longitudinal thrust N_{ox} has the constant value (-pR/2) throughout. Kendrick assumed the hoop thrust N_{oy} to have the constant value $-pRhL_f/(A_f+L_fh)$

throughout. This latter is an approximation since N_{oy} is really a function of x as can be seen from the theory of Von Sanden and Günther,[6] or from that of Salerno and Pulos,[8] but it can be shown that the error introduced by this approximation is negligible.

The extensional and bending strain energies, respectively, of a ring frame situated at a distance $x = rL_f$ from a bulkhead (one end of the cylinder) are given as follows:

$$(F_e)_r = \frac{EA_f}{2R}\int_0^{2\pi}\left[(w,_{\theta\theta}+w)\frac{\bar{e}}{R} - (v,_\theta - w)\right]^2_{x=rL_f} d\theta + \frac{N_{of}}{2R}\int_0^{2\pi}\left(u,_\theta^2 + w,_\theta^2 - 2v,_\theta w\right)_{x=rL_f} d\theta \tag{140}$$

$$(F_b)_r = \frac{EI_{xo}}{2R^3}\int_0^{2\pi}\left(w,_{\theta\theta}+w\right)^2_{x=rL_f} d\theta \tag{141}$$

where N_{of} is the hoop thrust in a ring frame just prior to buckling and will be assumed to have the value $-pRL_fA_f/(A_f+L_fh)$ instead of, say, that given by Equation (21). The quantity I_{xo} is the moment of inertia of a ring-frame section about the x-axis, and $\bar{e}$ is the eccentricity (+ inward) of the centroid of a ring-frame section.

The potential due to the radial pressure p, involving the additional displacements u, v, w, due to buckling is given by

$$W = -\frac{pR}{2}\int_0^{2\pi}\int_0^{L_b}\left|(2wu,_x + \frac{2}{R}wv,_\theta - v,_\theta u,_x - \frac{w}{R})^2\right. d\theta dx \tag{142}$$

The work done by the axial pressure p becomes equal to zero as a consequence of the trigonometric functions assumed for the buckling

displacements.

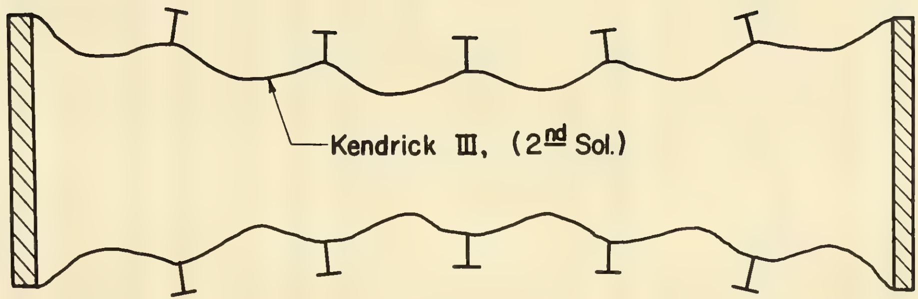
Kendrick attempted many solutions by using different buckling patterns in the energy integrals. However, the most plausible solution found and the one which has been confirmed by model tests[52] at the Model Basin is that based on the following buckling pattern used in the solution of Reference 51:

$$
\begin{aligned}
u(x,\theta) &= A_1 \cos n\theta \cos \frac{\pi x}{L_b} \\
v(x,\theta) &= B_1 \sin n\theta \sin \frac{\pi x}{L_b} + B_2 \sin n\theta \left(1-\cos \frac{2\pi x}{L_f}\right) \\
w(x,\theta) &= C_1 \cos n\theta \sin \frac{\pi x}{L_b} + C_2 \cos n\theta \left(1-\cos \frac{2\pi x}{L_f}\right)
\end{aligned}
\tag{143}
$$

Implicit in the above functions are the assumptions that the cylinder buckles into one-half sine wave from end to end, i.e., corresponding to simple-support conditions, and the interframe buckle shape is such that a zero-rotation condition exists at the ring frames, i.e., corresponding to clamped conditions. The radial component w of the buckling deformations defined by Equation (143) is shown in Figure 15.

Substituting Equation (143) into the energy integrals, Equations (138) through (142), the total potential energy given by Equation (137) becomes a function of the shell and frame geometry, the pressure loading, and the mode-shape parameters A_1, B_1, C_1, B_2, and C_2 of the buckling displacements. Hence, for a given geometry it can be seen that

$$V_T = V_T(A_1, B_1, C_1, B_2, C_2) \tag{144}$$



Radial Displacement Function, Equation (143)

$$w = C_1 \cos n\theta \sin \frac{\pi x}{L_b} + C_2 \cos n\theta \left(1 - \cos \frac{2\pi x}{L_f}\right)$$

Figure 15 - Longitudinal Configuration of Radial Buckling Displacement

The principle of minimum potential energy is invoked to determine the equilibrium state of the structure, for which the elastic energy V_T must have a stationary value. Any arbitrary variation from this value must vanish, and this leads to the following mathematical criterion:

$$\delta V_T = 0 = \frac{\partial V_T}{\partial A_1} \delta A_1 + \frac{\partial V_T}{\partial B_1} \delta B_1 + \frac{\partial V_T}{\partial C_1} \delta C_1 + \frac{\partial V_T}{\partial B_2} \delta B_2 + \frac{\partial V_T}{\partial C_2} \delta C_2 \qquad (145)$$

Now, since the variations δA_1, δB_1, δC_1, δB_2, and δC_2 in the buckling displacements are arbitrary so that they need not be zero, then, for equilibrium to exist, the following conditions must be satisfied simultaneously:

$$\frac{\partial V_T}{\partial A_1} = \frac{\partial V_T}{\partial B_1} = \frac{\partial V_T}{\partial C_1} = \frac{\partial V_T}{\partial B_2} = \frac{\partial V_T}{\partial C_2} = 0 \qquad (146)$$

The above conditions lead to a system of five linear, homogeneous, algebraic equations which must be solved simultaneously. In order that the nontrivial solution exist, which is $A_1 \neq B_1 \neq C_1 \neq B_2 \neq C_2 \neq 0$, the determinant formed by the coefficients of these shape parameters must vanish. This 5 X 5 stability determinant when expanded leads to a fifth-degree equation for the instability pressure. Extensive calculations conducted by Reynolds at the Model Basin with the aid of a UNIVAC computer have shown that the linearized form of the determinantal equation is more than adequate; the higher order terms in the pressure are almost insignificant. Reynolds developed a convenient graphical solution[53] of the Kendrick equations; he showed that the overall instability of a ring-stiffened cylinder can be expressed in a form similar to

Equation (136), that is, as the sum of two effects. One is a shell effect which reflects the membrane stiffness of the cylinder, as does the first term of Equation (57) for the panel-instability problem, except in the difference of the length (L_f vs L_b) of shell considered. The other is a frame effect which reflects the bending stiffness of the cylinder. Thus, Reynolds showed numerically that Kendrick's solution can be expressed in the form

$$p_{cr} = p_s\left(\frac{L_b}{R}, \frac{h}{R}, n\right) + p_f\left(\frac{I_e}{R^3 L_f}, n\right) \tag{147}$$

where the geometric parameters are the same as those for the Tokugawa-Bryant formula. Equation (147) further demonstrates the usefulness of the "split-rigidities" concept.

Ball[48] refined Reynolds' graphical solution and extended the range of usefulness by carrying out additional computer calculations for geometries of future interest. Figures 16 and 17 give the curves developed by Ball for both the Tokugawa-Bryant formula and the Kendrick solution. One point of interest to the designer is that these curves indicate the range of geometry for which disagreement exists between the two formulations. In such cases, it goes without saying that the curves corresponding to the Kendrick solution should be used.

Another approach to the general-instability problem was first proposed by Flügge; see Reference 9. He derived a set of differential equations for an orthotropic shell and showed how its solution could be used

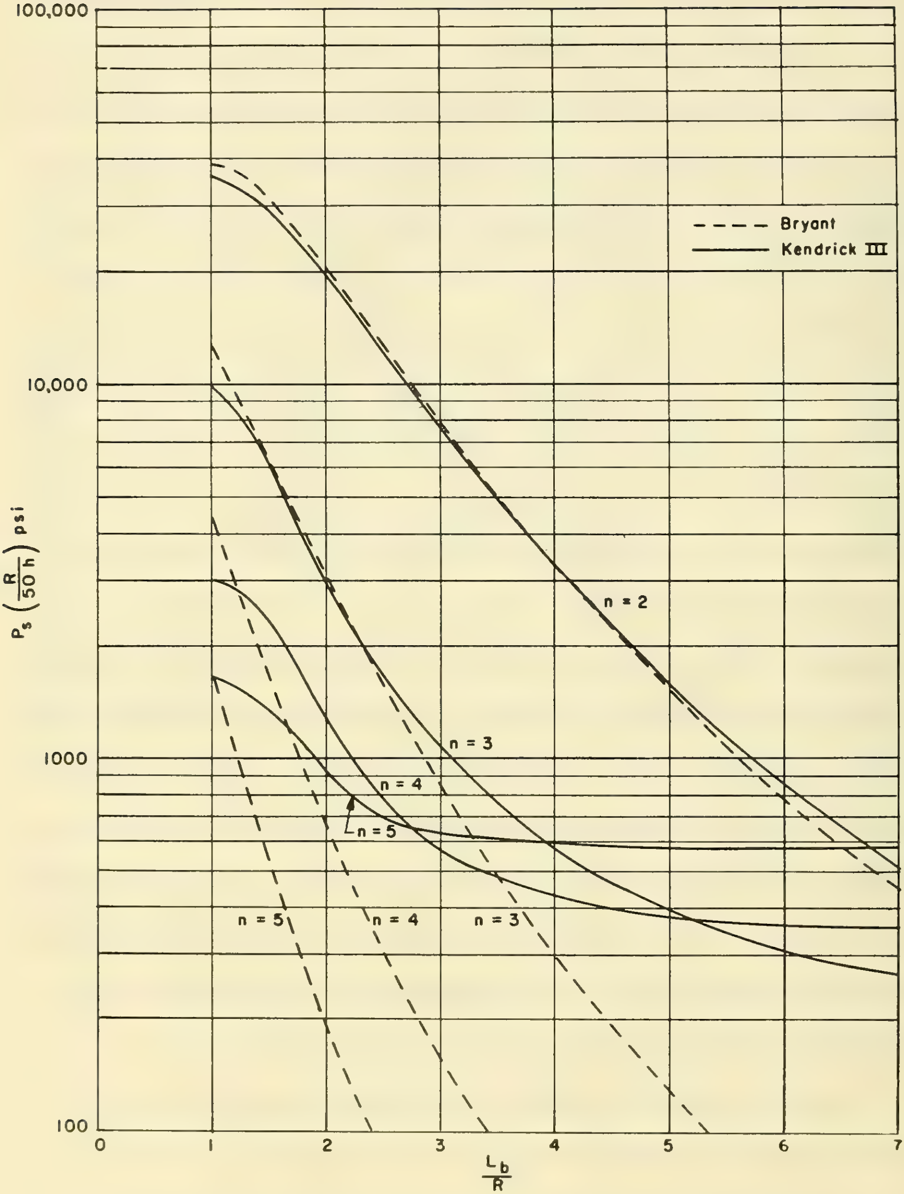


Figure 16 - Shell Pressure Factor p_s (for $E = 30 \times 10^6$ psi)

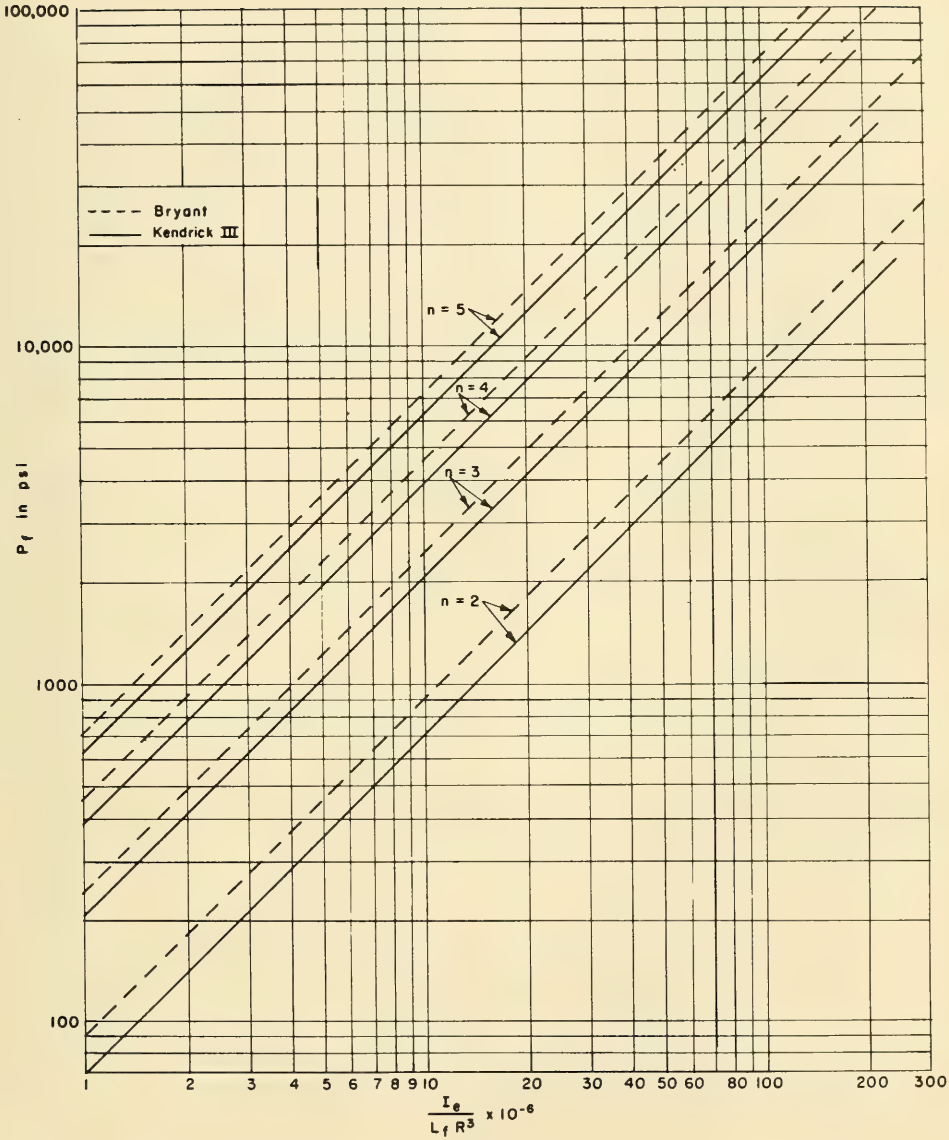


Figure 17 - Frame Pressure Factor p_f (for E = 30 x 10^6 psi)

for some special cases of stiffening. The basic idea is that the ring frames can be replaced by equivalent increases in the circumferential direction of the bending and extensional rigidities of the shell. If EI_e is the flexural rigidity of one ring frame together with an effective width of shell plating and L_f is the center-to-center distance between adjacent ring frames, we use EI_e/L_f instead of $Eh^3/12(1-\nu^2)$ in considering bending in the circumferential direction. An equivalent thickness $h_e = h(1 + A_f/L_f h)$, instead of h, is used in considering the hoop compression. Using the notations

$$\frac{I_e(1-\nu^2)}{L_f h R^2} = \alpha_1 \; ; \; \frac{h_e(1-\nu^2)}{h} = S \tag{148}$$

the equation for determining the critical values of general-instability pressure becomes

$$C_1 + C_2\bar{\alpha} + C_3\alpha_1 = C_4\phi_1 + C_5\phi_2 \tag{149}$$

in which $\bar{\alpha}$, ϕ_1, and ϕ_2 have the same meaning as given by Equation (53), and the coefficients C_1, C_2, ... etc. are as follows:

$$\begin{aligned} C_1 &= S\lambda^4 \\ C_2 &= \lambda^6(2n^2+\lambda^2) + S^2\lambda^2 n^2\left[2(\lambda^2-1)^2 + 2(n^2-1)^2 + 5\lambda^2 n^2 - 2\right] \\ C_3 &= (n^2-1)^2\left[\lambda^4 + S(n^2+2\lambda^2)n^2\right] \\ C_4 &= n^2\lambda^4 + S(n^2+2\lambda^2)n^4 - S(n^2+3\lambda^2)n^2 \end{aligned} \tag{150}$$

$$C_5 = \lambda^6 + S\lambda^2 n^2 (n^2 + 2\lambda^2 + 1) \tag{150}$$

$$\lambda = m\pi R / L_b$$

Note the similarity between Equations (149) and (55) and between the coefficients, Equations (150) and (56).

Bodner[54] working at the Polytechnic Institute of Brooklyn derived another set of equilibrium equations for an orthotropic cylindrical shell which are considerably simpler than those of Flügge. He used these equations to solve the case of a simply supported shell under hydrostatic pressure; and by relating the stiffness properties of the orthotropic shell to those of the ring-stiffened shell, he was able to derive a simple formula for the elastic general-instability pressure. For our purposes here, it suffices to give this final equation, which in the present notation takes on the following form:

$$p_{cr} = \frac{Eh}{R}\left(\frac{1}{n^2 + \lambda^2/2}\right)\left\{\frac{\lambda^4}{(n^2+\lambda^2)^2} + \frac{h^2\lambda^2}{6R^2(1-\nu^2)}\left(n^2 + \frac{\lambda^2}{2}\right)\right\} + \frac{n^4}{\left(n^2 + \frac{\lambda^2}{2}\right)} \frac{EI_e}{R^3 L_f} \tag{151}$$

where $\lambda = \dfrac{\pi R}{L_b}$. The similarity between Bodner's formula (151) and the Tokugawa-Bryant formula should be noted. The second term, that $\sim h^3$, of the so-called shell contribution to p_{cr} of Equation (151), is usually of minor importance so that the circumferential bending rigidity is predominately reflected by the frame term, i.e., the third term in Equation (151). In such a case, the Tokugawa-Bryant and Bodner formulas are almost identical in overall form except for two very important differences. These will now be discussed.

Since the differential equations used by Bodner to develop his orthotropic-shell solution were of the same approximation as Donnell's, this resulted in the approximation $n^2 - 1 \approx n^2$ in the denominator of both terms in Bodner's final formula (151). Basically, this approximation is in error because the general-instability buckling mode is associated with a small number of lobes, i.e., n is usually 2, 3, or 4, so that assuming n^2 to be very large in comparison to unity could lead to appreciable discrepancy. However, it turns out in numerical calculations that the frame term dominates, and although Bodner's coefficient

$$n^4 / (n^2 + \frac{\lambda^2}{2})$$

is different from $(n^2 - 1)$ of Equation (136), better agreement has been shown by Ball[48] to exist between pressures from Equation (151), than from Equation (136), with those computed using the solution of Kendrick. However, in the case of the infinite cylinder, i.e., $\lambda \rightarrow 0$, the Bodner frame coefficient reduces to n^2 instead of $(n^2 - 1)$. In this case, the Bodner formula gives pressures which are one third larger than the correct value given by the simple ring formula, i.e., $p = 3EI/R^3 L_f$.

Bijlaard[55] also developed an analysis for the general-instability mode of collapse for ring-stiffened cylinders using the method of "split rigidities." Also, a rather complete discussion of the orthotropic-shell approach to the problem of general-instability of stiffened cylinders for various loadings is given by Becker.[56] However, neither of these two

formulations are discussed further, nor are any of their final formulas included here because they involve certain parameters which are not too easily determined without having recourse to their original references.

INTERMEDIATE DEEP FRAMES TO INCREASE GENERAL-INSTABILITY STRENGTH

One of the more important geometric parameters which influences general-instability strength of ring-stiffened cylindrical pressure hulls is the overall length L_b, which, in actuality, is the distance between rigid holding bulkheads in submarine construction. Certain designs, and this is becoming more common, particularly for deep-submergence vehicles, preclude the use of rigid internal bulkheads to break up the overall length of long hull compartments. The requirements may, for example, stem from internal arrangements and other space considerations. In such cases, the most efficient manner in which the general-instability strength of long cylindrical compartments can be increased without increasing overall shell thickness and/or cross-sectional properties of all the ring frames, with their concomitant prohibitive weight increase, is the use of intermediate heavy frames or mixed-framing arrangements. Such stiffening systems can provide large increases in general-instability strength by effectively "breaking up" these long hull compartments. In order to efficiently design for maximum possible instability strength of such structures within certain specified weight limitations, it is necessary to have adequate theory which, when verified by experiment, can then be converted to design criteria.

The first attempt at a rigorous mathematical solution of the elastic general-instability problem for long cylindrical shells, which are stiffened by a uniform distribution of typical light ring frames closely spaced, and a set of two or more, depending on the length L_b, intermediate heavy ring frames widely spaced apart, is that attributed to Kendrick[57] of the Naval Construction Research Establishment. Later, Reynolds[58] at the Model Basin discovered certain shortcomings in the buckling functions assumed by Kendrick, and he revised the original analysis to conform more realistically to the experimental observations reported in Reference 58.

The basic approach used in the mathematical formulations by both Kendrick and Reynolds was that of using the energy method and writing expressions for the elastic strain energies for the shell, the typical light ring frames, and the intermediate heavy ring frames. The procedure is almost identical to that followed in connection with Equations (137) through (142), except that the ring energy terms

$$\sum_{r=1}^{N}\left(F_e\right)_r \text{ and } \sum_{r=1}^{N}\left(F_b\right)_r$$

in Equation (137) must be rewritten as follows to include the strain energies of the heavy frames:

$$V_{rings} = \sum_{r=1}^{(N_1+1)(N_2+1)-1}\left(F_e+F_b\right)_f - \sum_{R=1}^{N_2}\left(F_e+F_b\right)_f + \sum_{R=1}^{N_2}\left(F_e+F_b\right)_F \qquad (152)$$

where

N_1 is the number of light ring frames,

N_2 is the number of heavy ring frames, and the subscripts f and F refer to light and heavy ring frames, respectively.

The original ring-energy integrals, Equations (140) and (141), as used for the present problem were modified by Kendrick to include some additional terms as follows:

$$F_e = \frac{EA}{2R}\int_0^{2\pi}\left[\left(w,_{\theta\theta}+w\right)\frac{\bar{e}}{R}-\left(v,_\theta - w\right)\right]_x^2 d\theta + \frac{N_o}{2R}\int_0^{2\pi}\left\{\left[\left(u,_\theta^2+w,_\theta^2-2v,_\theta w\right)\left(1+\frac{\bar{e}}{R}\right)+\right.\right.$$

$$\left.\left. + 2v,_\theta\left[\left(v,_\theta - w\right)-\left(w,_{\theta\theta}+w\right)\frac{\bar{e}}{R}\right]\right]\right\}_x d\theta \tag{153}$$

$$F_b = \frac{EI}{2R^3}\int_0^{2\pi}\left(w,_{\theta\theta}+w\right)_x^2 d\theta \tag{154}$$

Reynolds pointed out certain shortcomings in regard to the hoop thrusts in the ring frames just prior to buckling as assumed by Kendrick, and the following expressions have been used in the revised theory:

$$N_{OF} = -\frac{pRA_F L_f}{A_f + L_f h} \tag{155}$$

$$N_{of} = -\frac{pRA_f L_f}{A_f + L_f h} \tag{156}$$

for the heavy and light frames, respectively, and where A_F and A_f are the cross-sectional areas of the heavy and light frames, respectively.

The ring-energy integrals, Equations (153) and (154), are evaluated at $x = rL_f$ for the first summation of Equation (152), and at $x = RL_F$ for the second and third summations of Equation (152).

The shell-energy integrals, Equations (138) and (139), remain the same for the present problem and so do the membrane forces in the shell prior to buckling, i.e.,

$$N_{ox} = -\frac{pR}{2} \tag{157}$$

$$N_{oy} = -\frac{pRL_f h}{A_f + L_f h} \tag{158}$$

The justification for using Equations (155), (156), and (158) instead of the analogous expressions of Kendrick is discussed in Reference 58.

With all this, the total potential of the system comprised of the shell and ring strain energies and the work done by the external loads is given by

$$V_T = U_e + U_b + V_{rings} + W \tag{159}$$

where U_e, U_b, V_{rings}, and W are given by Equations (138), (139), (152), and (142), respectively.

The assumed buckling displacements used by Kendrick[57] for the present problem are given by the following expressions:

$$\begin{aligned} u(x,\theta) &= A_1 u_1(x) \cos n\theta \\ v(x,\theta) &= \left[B_1 v_1(x) + B_2 \left|\sin\frac{\pi x}{L_f}\right| + B_3 \left|\sin\frac{\pi x}{L_f}\right|\right] \sin n\theta \end{aligned} \tag{160}$$

$$w(x,\theta) = \left[C_1 w_1(x) + C_2\left|\sin\frac{\pi x}{L_f}\right| + C_3\left|\sin\frac{\pi x}{L_f}\right|\right]\cos n\theta \qquad (160)$$

where

$$\left.\begin{aligned} u_1(x) &= \cos\frac{\pi x}{2qL_f} \\ v_1(x) = w_1(x) &= \sin\frac{\pi x}{2qL_f} \end{aligned}\right\} 0 \le x \le qL_f \qquad (161)$$

$$\left.\begin{aligned} u_1(x) &= 0 \\ v_1(x) = w_1(x) &= 1 \end{aligned}\right\} qL_f \le x \le (L_b - qL_f) \qquad (162)$$

$$\left.\begin{aligned} u_1(x) &= \cos\left[\pi\frac{(x-L_b+2qL_f)}{2qL_f}\right] \\ v_1(x) = w_1(x) &= \sin\left[\pi\frac{(x-L_b+2qL_f)}{2qL_f}\right] \end{aligned}\right\} (L_b-qL_f) \le x \le L_b \qquad (163)$$

where

x, θ	are the axial and circumferential coordinates, respectively
u, v, and w	are the axial, tangential, and radial displacements, respectively
$A_1, B_1, \ldots$ etc.	are arbitrary coefficients
L_b	is the bulkhead spacing, and
q	is an arbitrary number which permits the formation of a straight central portion of variable lengths as shown in Figure 18.

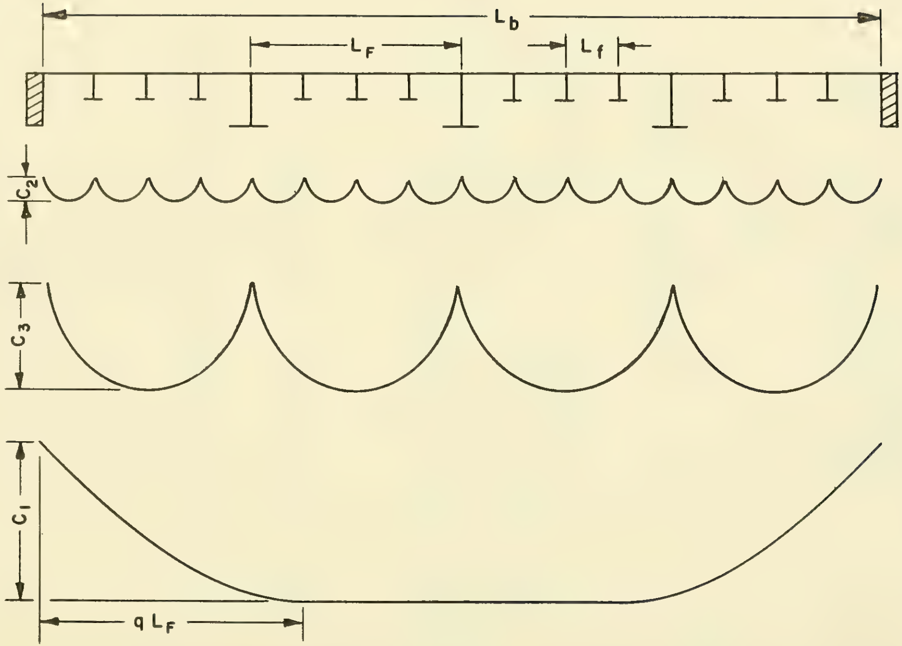


Figure 18 - Longitudinal Configuration of Radial Buckling Displacement for Stiffened Cylinder with Intermediate Heavy Frames

When q is given the value $\frac{L_b}{2L_f}$, the shape of the radial deflection w between bulkheads is a half-sine wave. Since there are seven arbitrary coefficients in the buckling deformations, Equation (160), the system possesses, in a sense, seven degrees of freedom which allow independent deformations between the light frames, the heavy frames, and the ends of the cylinder; the three degrees associated with the w component of displacement are shown in Figure 18.

Reynolds pointed out that the generality of the analysis could be improved by permitting one additional degree of freedom. This was done by adding to the axial displacement $u(x,\theta)$ a second component varying periodically between adjacent heavy frames, i.e., Reynolds suggested the following function instead of that given in Equation (160):

$$u(x,\theta) = \left[A_1 u_1(x) + \frac{L_F}{\pi} A_3 \frac{d}{dx}\left|\sin\frac{\pi x}{L_F}\right|\right]\cos n\theta \qquad (164)$$

This is discussed further in Reference 58.

The method of solution then goes along the following lines: the assumed buckling displacements, Equations (160) and (164), are substituted into the integrals for the shell and ring energies and for the work done, and thus the total potential, Equation (159), is determined. The condition of minimum potential energy, i.e., $\delta V_T = 0$ (see Equation (145), for example), leads to a system of eight linear homogeneous algebraic equations for the eight coefficients A_1, A_3, B_1, B_2, B_3, C_1, C_2, and C_3. The 8 X 8 stability determinant formed by the coefficients of the A's, B's, and C's when

expanded leads to an equation from which the buckling pressures can be determined. This system of equations has been programmed on the Model Basin IBM-7090 computer for purposes of numerical computations and verification of the analysis.

In lieu of this rather complicated mathematical solution, which is necessary for a complete analysis and insight into the phenomenologic aspects of the problem, Blumenberg[58] developed some convenient empirical formulas for design purposes on the basis of test data he and Reynolds obtained. He adapted the well-known Lévy ring formula to the present problem by redefining the geometric parameters as follows:

$$I_{FS} = \left(\frac{p_{cr} R_c^3 L_e}{(n^2-1)E}\right) \tag{165}$$

where it is assumed that

$$L_e \approx L_f + (L_F - L_f)\left(\frac{p_{cr} - p_B}{p_x - p_B}\right) \tag{166}$$

for the range where $p_x \geq p_{cr} \geq p_B$. Also, the following additional definitions are used:

I_{FS} is the moment of inertia of the heavy-frame-shell section.

p_{cr} is the critical pressure of the actual cylinder including all the frames.

R_c is the radius from the cylinder axis to the centroid of the heavy-frame-shell section.

L_e is the effective length of that portion of the cylinder which loads the heavy frame.

p_B is the critical pressure of the cylinder with the heavy frames replaced with typical light frames (see Figure 19).

n is the critical buckling mode associated with p_B.

L_f is the spacing between typical light frames.

L_F is the spacing between intermediate heavy frames.

p_x is equal to p_F or p_n, whichever is lower.

p_F is the critical pressure of the uniformly stiffened cylinder of length L_F (see Figure 19).

p_n is the critical pressure at which the critical mode changes from n to (n+1) as the length of the uniformly stiffened cylinder is reduced (see Figure 19).

Thus, the size of the heavy frame in the pressure range $p_x \geq p_{cr} \geq p_B$ is dependent upon two limiting conditions. For the lower limit p_B, the heavy frame is equal in size to a typical light frame and the load acting on it is the pressure over one typical frame spacing of shell. As the heavy frame is made larger, it assumes increasingly more of the total load. At the upper pressure limit p_x, the maximum pressure for which there exists an overall symmetrical buckling shape in the longitudinal direction, the heavy frame is loaded by the pressure acting on one heavy-frame spacing of cylinder.

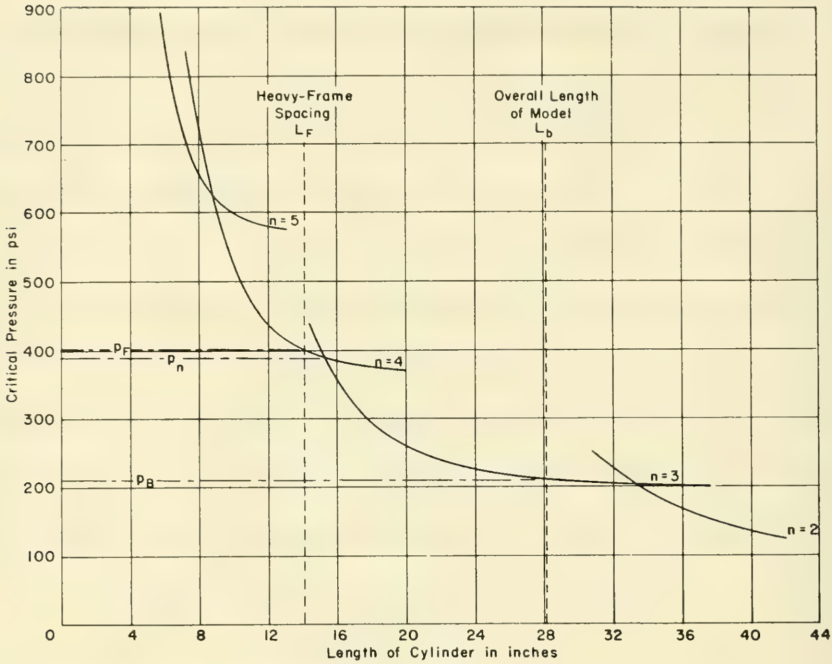


Figure 19 - Kendrick Part III (Second Solution)[51] versus Length of Uniformly Stiffened Cylinder

The size of the heavy frame in the pressure range $p_F \geq p_{cr} \geq p_n$ can be calculated from the formula

$$I_{FS} = I_n + (I'_{FS} - I_n)\left(\frac{p_{cr} - p_n}{p_F - p_n}\right) \tag{167}$$

where

$$I'_{FS} = \frac{p_F R_c^3 L_F}{3E} \tag{168}$$

and I_n is the moment of inertia of the heavy-frame-shell section determined for p_n from Equation (165). Any further increase in the strength of the heavy frame will not increase the critical pressure because the failure will occur between the heavy stiffeners.

In Reference 58, the authors compare calculations using Equations (165) and (167) with observations from model tests. On the basis of the good agreement obtained between prediction and measurement, they suggest that the pressures p_B and p_F be computed using the Tokugawa-Bryant formula, Equation (136), and that p_n be computed using the following formula:

$$p_n = \frac{n(n+2)EI_{fs}}{R_c^3 L_f} \tag{169}$$

In this way, the need for drawing curves similar to those of Figure 19 is eliminated. Figure 19 shows the results of calculations for a specific geometry considered by Blumenberg and Reynolds as part of their model test program. The pressures p_B, p_F, and p_n entering into Equations (166) and (167) are shown for this case.

ELASTIC DEFORMATIONS AND STRESSES IN IMPERFECTLY CIRCULAR RING FRAMES

The theories and formulas presented so far for predicting the elastic general-instability strength of ring-stiffened cylindrical pressure hulls have all been based on the assumption that the structure is initially perfectly circular. In the fabrication of submarine hulls, it invariably turns out that due to the cold-forming process and welding of steel plating into

the cylindrical form, the hull structure is far from the ideal case assumed in the theory.

The problem of the additional elastic stresses due to imperfect circularity of the shell alone and their influence on collapse strength has already been considered in the discussion of the Galletly-Bart[44] and Bodner-Berks[45] formulations. These analyses were based on the assumption that the ring frames were initially perfectly circular and remained so during the buckling and collapse stages of the shell structure. For the buckling strength of the overall cylinder structure, the circularity and the state-of-stress in the ring frames is of paramount importance because local failure of the ring frames could precipitate a premature overall collapse just as if no stiffening existed. The ring frames are also intended to provide adequate circularity to the shell portion. Pulos and Hom[59] working at the Model Basin have discussed the basic mechanism by which the local circumferential bending stresses induced in the ring frames by out-of-roundness when superposed on the axisymmetric compressive stress due to the pressure loading can cause premature yielding thus leading to failure; see Reference 60 for the results of some model tests using deliberately out-of-round stiffened cylindrical hulls.

Kendrick[61] working at the Naval Construction Research Establishment developed an analysis for determining the maximum stresses in the transverse stiffeners of imperfectly circular cylindrical shells. This frame out-of-roundness theory followed closely his earlier work for

determining the elastic general-instability pressures of a perfect cylinder. The basic approach was again the energy method in which an out-of roundness and a buckling pattern are assumed, and the principle of minimum potential energy is invoked to determine the amplitude coefficients. In his analysis, Kendrick assumed that the out-of-roundness coincides with the buckling shape associated with the lowest elastic general-instability pressure of the perfect cylinder; that is, the out-of-roundness function is sinusoidal in both the circumferential and longitudinal directions, i.e.,

$$w_o(x,\theta) = C_o \cos n\theta \sin \frac{m\pi x}{L_b} \tag{170}$$

where w_o is the radial deviation from perfect circularity and C_o is the amplitude of this deviation.

Since in actual construction of reinforced cylindrical pressure hulls, the initial overall longitudinal "sagging" implied by Equation (170) is highly improbable, Hom[60] working at the Model Basin devised a new analysis based on the following more realistic out-of-roundness shape:

$$w_o(\theta) = C_o \cos n\theta \tag{171}$$

This function implies that the cylinder generators over the bulkhead spacing L_b are straight and parallel but that the circumferential profile of the shell varies from the perfect circle in a sinusoidal manner.

The mathematical techniques used by both Kendrick and Hom were almost identical, and they have already been discussed in connection with

the panel-buckling and general-instability problems. Both investigators used the following buckling deformations:

$$\bar{u}(x,\theta) = A \cos n\theta \sin \frac{m\pi x}{L_b}$$
$$\bar{v}(x,\theta) = B \sin n\theta \sin \frac{m\pi x}{L_b} \qquad (172)$$
$$\bar{w}(x,\theta) = C \cos n\theta \sin \frac{m\pi x}{L_b}$$

i.e., it is assumed that the cylinder will buckle into m half waves in the longitudinal direction and n full waves in the circumferential direction. Further, the ends of the cylinder are assumed to be simply supported.

The energy integrals for the shell and the ring frames and the integral for the work done (used by both investigators) were somewhat more complicated than Equations (138) through (142) because of additional terms that enter as a consequence of the out-of-roundness w_o. These integrals are not given here, but the reader is referred to Equations (4) through (8) in Reference 60. Once the total potential energy of the elastic system (see Equation (137)) is determined, the minimum energy criterion leads to a system of three linear, nonhomogeneous, algebraic equations which must be solved simultaneously; these are of the following form:

$$a_{11}A + a_{12}B + a_{13}C = -a_{14}C_o$$
$$a_{12}A + a_{22}B + a_{23}C = -a_{24}C_o \qquad (173)$$
$$a_{13}A + a_{23}B + a_{33}C = -a_{34}C_o$$

where a_{11}, a_{12}, a_{13}, ... etc. are rather complicated expressions of the geometry and loading; see Equations (27) in Reference 60. From Equations (173), the displacement coefficients A, B, and C may be determined in terms of the out-of-roundness amplitude C_o. The case $C_o = 0$ leads us back to the eigenvalue problem of the elastic general instability.

Once the coefficients A, B, and C are determined in terms of C_o, the elastic stresses due to imperfect circularity, at any point on the periphery of the frame flange, can be found in terms of the amplitude C_o from the equation

$$\sigma_f = -E\left[(\bar{w} - \frac{\partial \bar{v}}{\partial \theta})\frac{1}{R} + \left(\frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial \bar{v}}{\partial \theta}\right)\frac{e_f}{R^2}\right]\left(\frac{R}{R-e_f}\right) \qquad (174)$$

where the displacements $\bar{u}$, $\bar{v}$, and $\bar{w}$ are given by Equation (172); Equation (174) is derived in Reference 60. The first term in the brackets of Equation (174) represents a direct-stress component whereas the second term represents a bending component; both components are due to the asymmetric bending action as a consequence of the out-of-roundness. To obtain the total stress in the flange of a ring frame, the axisymmetric membrane component due to the radial load Q^* from Equation (21) must be added to that of Equation (174); this latter stress is given by

$$\sigma_{\phi f} = \frac{Q^* R^2}{(A_{eff} + bh)(R + d + \frac{h}{2})} \qquad (175)$$

where d is the depth of the circular ring frame. Equation (175) is valid for external frames; for internal frames, the factor $(R - d - \frac{h}{2})$ must be used in the denominator (see Figure 11).

Numerical calculations using Equations (173) are facilitated with the aid of a digital computer. Their solution represents an "exact" solution of the problem within the framework of thin-shell assumptions and the assumed out-of-roundness and buckling functions. However, both Kendrick and Hom developed so-called approximate solutions from these more exact equations; these latter are proving useful in design. Hom gives the following convenient formula for determining the maximum bending stress in the flange of an imperfectly circular ring frame:

$$\sigma_{bf} = \pm \frac{4}{\pi}\left[\frac{E e_f C_o}{R^2}(n^2 - 1)\left(\frac{p}{p_{cn} - p}\right)\right] \tag{176}$$

where e_f is the distance of the frame flange from the median surface of the shell and is positive for internal frames and negative for external frames.

Note should be taken of the magnification term $(\frac{p}{p_{cn}-p})$ in Equation (176), which is analogous to that of Equation (127) for the shell out-of-roundness problem. These factors suggest that initial imperfections "grow" in a nonlinear fashion with the applied static pressure p. This phenomenon is similar to the resonance condition in vibration problems when the frequency of an applied force approaches the natural frequency of the structure. In the static pressure problem of interest to us here, "resonance" occurs when the applied pressure p approaches the elastic general-instability pressure p_{cn} of the perfectly circular ring-stiffened cylinder. The pressures p_{cn} can be conveniently determined from the curves given by Ball in Reference 48 and reproduced in this report as Figures 15 and 16.

In Reference 60 Hom presents experimental data from tests of structural models which have been purposely fabricated imperfectly circular in specified out-of-roundness patterns. He also presents a comprehensive comparison and discussion of theory vis-à-vis experiment.

PLASTIC GENERAL INSTABILITY OF SHELL AND RING FRAMES

Although it is true that prior to reaching its ultimate load-carrying capacity, a ring-stiffened cylindrical shell first undergoes elastic deformations, optimum design of such a structure must eventually be based on considerations of inelastic behavior. The elastic general-instability mode and the theories developed to predict this behavior have already been discussed in a previous section. It remains for us to give due consideration to the inelastic general-instability mode as was done for both the axisymmetric and asymmetric inelastic panel-instability modes.

Taking advantage of what has already been discussed in connection with the asymmetric inelastic buckling of a stiffened cylindrical shell between adjacent ring frames, and coupling this to the "split-rigidities" concept underlying the Tokugawa-Bryant formula for elastic general instability, it becomes rather obvious to the reader what a possible approach to the inelastic general-instability problem might be.

Lunchick[62] followed this avenue of thinking, and by introducing the Shanley tangent-modulus concept[63] for the buckling of a column in the inelastic range, he was able to derive a formula for plastic general insta-

bility of ring-stiffened cylindrical shells. It can be shown that this formula may be expressed as follows:

$$p_c = \frac{E_s h}{R}\left\{\frac{\lambda^4}{\left(n^2-1+\frac{\lambda^2}{2}\right)\left(n^2+\lambda^2\right)^2\left[1-\frac{3}{4}\left(1-\frac{E_s}{E_t}\right)\frac{\lambda^4}{\left(n^2+\lambda^2\right)^2}\right]}\right\}+\left(n^2-1\right)\frac{E_t I_e}{R^3 L_f} \qquad (177)$$

where $\lambda = \frac{\pi R}{L_b}$.

It is instructive for us to compare Equations (177) and (136). The form of the first term, the so-called shell contribution, in each of these two formulas is identical except for two differences which reflect the difference between inelastic and elastic behavior, respectively; first, the elastic modulus E of Equation (136) is replaced by the secant modulus E_s in Equation (177); and second, the denominator term of Equation (177) has a multiplying factor in brackets which is exactly the same as the second multiplying factor in brackets in the denominator of Equation (104). The form of the second term, the so-called frame contribution, in each of Equations (177) and (136) is identical with one exception: the elastic modulus E of Equation (136) is replaced by the tangent modulus E_t in Equation (177). This latter stems from the replacement of the elastic modulus in the column-buckling equations with the tangent modulus to generalize the Euler formula so that it applies in both the elastic and inelastic ranges; this is due to Shanley.[63]

Since Equation (177) can be easily deduced from what has already been developed in the preceding sections, and from what has been said above, details of its development will not be given here. Investigations

are presently underway at the Model Basin to obtain some verification of Equation (177).

The method for determining the secant and tangent modulii E_s and E_t, respectively, from the uniaxial stress-strain curve of a given material comprising the shell structure and for a given state of stress defined by σ_i (see Equation (107)) has already been outlined in connection with the axisymmetric and asymmetric panel-instability modes. The same techniques are also used here, and the procedure for finding the buckling pressure in the plastic range follows the method shown on Figure 14.

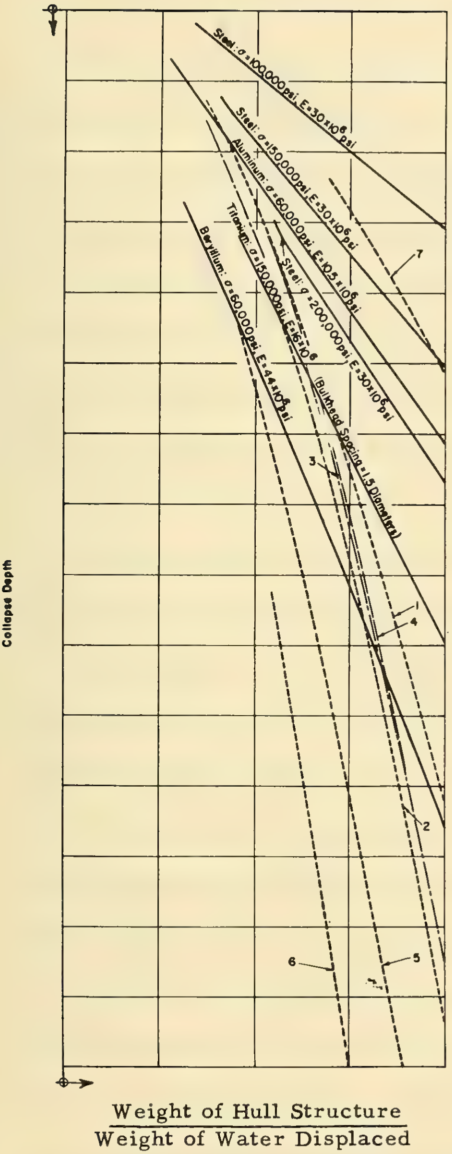
SOME REMARKS ON NEW PRESSURE HULL STRUCTURES FOR DEEP DEPTH

In the preceding sections, consideration was given to the more important physical concepts and mathematical analyses, and the equations and formulas resulting therefrom, which today form the basis for rational design of cylindrical pressure hull structures. The question as to how these formulations can be used collectively in an optimum design procedure is left to the discretion of the reader. However, it goes without saying that the most obvious approach would be to program the various equations and formulas for high-speed digital computation. It is then possible that for specified mechanical properties of a hull material, the pressure and stresses associated with each of the primary modes of failure can be determined for a broad range of the geometric parameters of interest. It then remains to optimize, by some semigraphical technique for instance, the best possible configurations for a given weight-displace-

ment or buoyancy ratio. Once a particular geometry is chosen, it then remains to carry out detailed buckling and stress analyses to check the adequacy of the design.

The results of such an attempt at optimization may look like the strength-weight curves presented by Buhl, Pulos, and Graner in Reference 64; these are reproduced here as Figure 20. It is rather obvious from these curves all other considerations like fabricability, creep resistance, fatigue strength, etc. being equal, what the strength potential of various hull materials appears to be. Some general discussion of the materials problem as such and the research programs underway to find the necessary answers is given by Owen and Sorkin in Reference 5. Later investigators have given some detailed consideration to both the advantages and some of the apparent shortcomings of specific hull materials. Sorkin and Willner[65] have examined the high-strength titanium alloys as possible hull structural materials, and Buhl, Pulos, and Graner[64] examined certain possibilities offered by the fiber-reinforced plastics.

In Reference 3, Wenk also presents the results of strength calculations he conducted for different hull materials and for a wide range of geometry. However, his curves are primarily of qualitative value because he introduced a number of questionable approximations and simplifications to make the numerical computations tractable; it can also be said that some of the design equations and criteria he used are not the most up to date. The same is true of a more recent paper by Gerard[66] on the minimum-



Fiberglass Characteristics

Curve	Material	Density, lb/in^3	σ, psi	E, psi	Bulkhead Spacing / Diameter
1	Glass-Fiber Reinforced Plastic	0.07	70,000	4.8×10^6	1.5
2		0.07	100,000	4.8×10^6	1.0
3		0.075	90,000	6.0×10^6	1.5
4		0.08	100,000	10×10^6	2.0
5		0.08	150,000	10×10^6	1.5
6		0.08	200,000	10×10^6	1.0
7	Steel-Filament Reinforced Plastic	0.186	150,000	12×10^6	1.5

Figure 20 - Strength-Weight Characteristics of Ring-Stiffened Cylindrical Pressure Hulls Made of Different Materials

weight design of ring-stiffened cylinders under external pressure.

One of the basic assumptions underlying all the mathematical formulations presented in the main portion of this paper is that associated with thin-shell theory. Immediately, the question arises as to what is the limit of applicability of the equations and formulas which have been presented. If we accept the thesis of Novozhilov,[67] then it appears that the approximations of thin-shell theory may introduce errors on the order of 5 percent for thickness-radius ratios of about 1/20 for cylindrical pressure hulls. This small magnitude of error provides the designer a great deal of flexibility because the wall thicknesses required for pressure-hull structures to withstand the pressures for operation at great depths, even those depths covering 98 percent of the world's oceans, may still be no thicker than the 1/20 ratio so that the equations we have set forth can provide adequate solutions. Of course, this is also contingent on the hull material used, which, in turn, influences the thickness, but in general, it can be said that probably the upper limit on the error introduced by using thin-shell theory should be less than 10 percent.

The assumptions of thin-shell theory have been examined by Klosner and Kempner[68] in light of results they found from a three-dimensional elasticity solution for the case of a long thick cylinder under the action of a single radial band load around its periphery. These investigators concluded that the classical shell theories of Timoshenko and Flügge represent good approximations to the three-dimensional stress problem

for relatively thick shells, e.g., $R_i/R_o = 0.9$ and $h/R = 1/9.5$, (where R_i is the radius to the inside surface of the cylinder, R_o is that to the outside surface, R is that to the median surface, and h is the thickness of the cylinder). However, they found that these shell theories do not adequately predict the stresses and radial deflections in the neighborhood of concentrated loads. At these locations, the radial deflections are predicted quite accurately by the transverse shear deformation shell theories. Interestingly enough, shear deformation theory was found not to improve the accuracy of the axial displacements and the stresses.

For the particular problem of a long circular cylindrical shell of constant thickness subjected to a radial line load considered by Klosner and Kempner, the authors found that the classical shell theories predict deflections which are 8 percent smaller than those obtained from the elasticity solution. Thus, if one would consider a ring-stiffened cylindrical shell subjected to a uniform pressure, then the maximum error in the calculated interaction load may be as great as 8 percent and will occur when the spacing of the reinforcing rings is large and the rings are rigid. As the rigidity of the ring frames decreases, the error decreases. Thus, for ordinary pressure-hull design, the interaction loads are not significantly different whether use is made of classical shell theories or three-dimensional elasticity or shear deformation shell theories. What is significant, however, is the difference in the stress distribution, and it is with this consideration in mind that future work pertaining to pressure-hull

design should be directed. Of course, if the shell is much thicker than $R_i/R_o = 0.9$, then the interaction loads will not be adequately predicted by the use of a classical shell theory.

The problem of predicting the ultimate load-carrying capacity of a thick-walled pressure hull may be somewhat simpler than that associated with the shallower depth designs. This latter question has been adequately answered in an earlier section and the pertinent formulas used for predicting axisymmetric collapse, of relatively thin cylinders, precipitated by yielding have been given as Equations (29), (31), (32), (35), (36), ... etc. For the case of the thicker walled hulls required to withstand the greater pressures of deep depth, the following simple equations provide the necessary means for predicting collapse strength:

$$\bar{\sigma}_x = \frac{pR_o}{2h}\,;\ \bar{\sigma}_\phi = \frac{pR_o}{h(1+A_f/L_f h)}\,;\ \bar{\sigma}_r = \frac{pR_o}{(R_o+R_i)} \qquad (178)$$

The stresses given by Equations (178) appropriately define the three-dimensional state of stress in the shell wall; the stress $\bar{\sigma}_x$ results from considerations of equilibrium in the axial direction, whereas the stresses $\bar{\sigma}_\phi$ and $\bar{\sigma}_r$ in the circumferential and radial directions, respectively, result from integrating the Lamé[17] stresses through the thickness of the shell. The added thickness A_f/L_f reflected by the term in the denominater for $\bar{\sigma}_\phi$ represents the stiffening action of the ring frames and is a consequence of "spreading" the frame area A_f out over a frame spacing L_f. The stresses, Equation (178), can then be used in conjunction with the Huber-

Hencky-Von Mises failure criterion[17] to predict the pressure at which the "effective stress" σ_i reaches the yield strength σ_y of the material. The appropriate collapse criterion can be derived from

$$\frac{1}{\sqrt{2}}\left[(\bar{\sigma}_x - \bar{\sigma}_\phi)^2 + (\bar{\sigma}_\phi - \bar{\sigma}_r)^2 + (\bar{\sigma}_r - \bar{\sigma}_x)^2\right]^{1/2} \equiv \sigma_y \qquad (179)$$

so that once the stresses, Equation (178), are substituted into Equation (179), the following expression for the collapse pressure is obtained:

$$p_c = \frac{\sigma_y h/R_o}{\left\{\frac{1}{4} + \frac{1}{1+\alpha}\left[\frac{1}{1+\alpha} - \bar{t} - \frac{1}{2}\right] + \bar{t}\left[\bar{t} - \frac{1}{2}\right]\right\}^{1/2}} \qquad (180)$$

where

$$\alpha \equiv \frac{A_f}{L_f h}\,;\; h \equiv R_o - R_i\,;\; \bar{t} \equiv \frac{R_o - R_i}{R_o + R_i} \qquad (181)$$

If we accept thin-shell theory as a valid starting point, then it appears that the choice of thicker walled pressure-hull structures of the future will depend to a large measure on the ease with which they can be economically fabricated. In cases of hulls requiring thicknesses of plating up to, say, about 6 inches, and/or materials which are not easily fabricable and weldable, the conventional construction techniques associated with the monolithic cylindrical shell stiffened by the usual type of transverse ring-frames is no longer practical. Recourse must therefore be had to new and, as yet, untried hull concepts and constructions. It is highly possible that the pressure-hull structure of the future will look like the configuration shown in Figure 21. This shows the composite or sheathed concept applied to either a conventional ring-stiffened cylinder or to a sandwich cylinder.

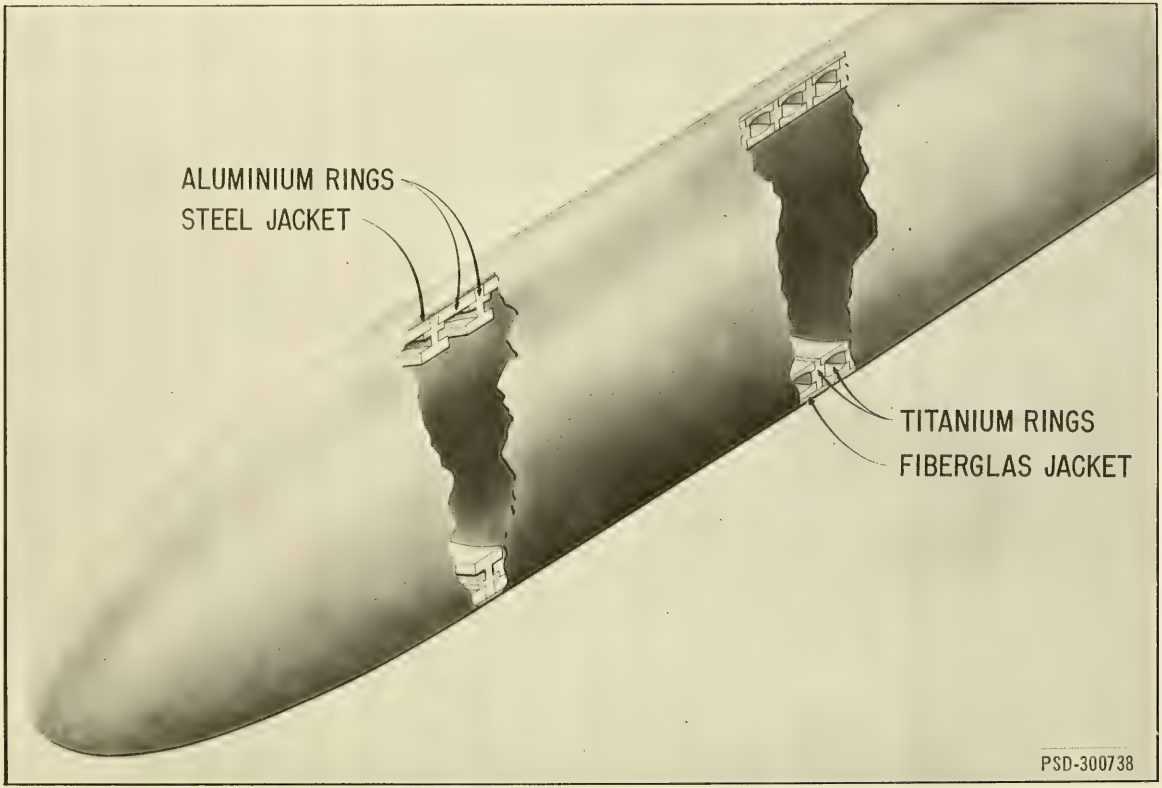


Figure 21 - Composite Cylindrical Pressure Hull

As part of its broad program of structural research into novel hull structures, in conjunction with some of the newer high-strength materials such as the HY-steels, the HY-titanium and aluminum alloys, and even the fiber-reinforced plastics, the Model Basin is examining a broad spectrum of the more promising geometrical configurations and construction ideas. These include stiffened and unstiffened spheres, prolate spheroids, and other shells of double curvature; bimetallic and other composite and sheathed arrangements; sandwich construction; multilayered cylinders, and multilayered spheres; "membrane" shells; and metal tape-wound and filament-wound cylinders. These efforts include both analytical studies and structural model tests. A number of these problem areas are also being investigated by organizations outside the United States Navy, but these studies are primarily under Bureau of Ships and Office of Naval Research sponsorship.

A complete discussion of the state of knowledge for these new hull concepts is a presentation in itself so that it is beyond the scope of this report. However, it is pertinent that some introductory remarks be made on the subject.

Let us begin by saying that in the range of the thicker hull structures required to withstand the greater pressures of operation, it can be shown that with judicious design evolved from rigorous mathematical theory, there is probably very little difference in structural efficiency between the unstiffened, the ring-stiffened, and the sandwich cylinder. This fact is

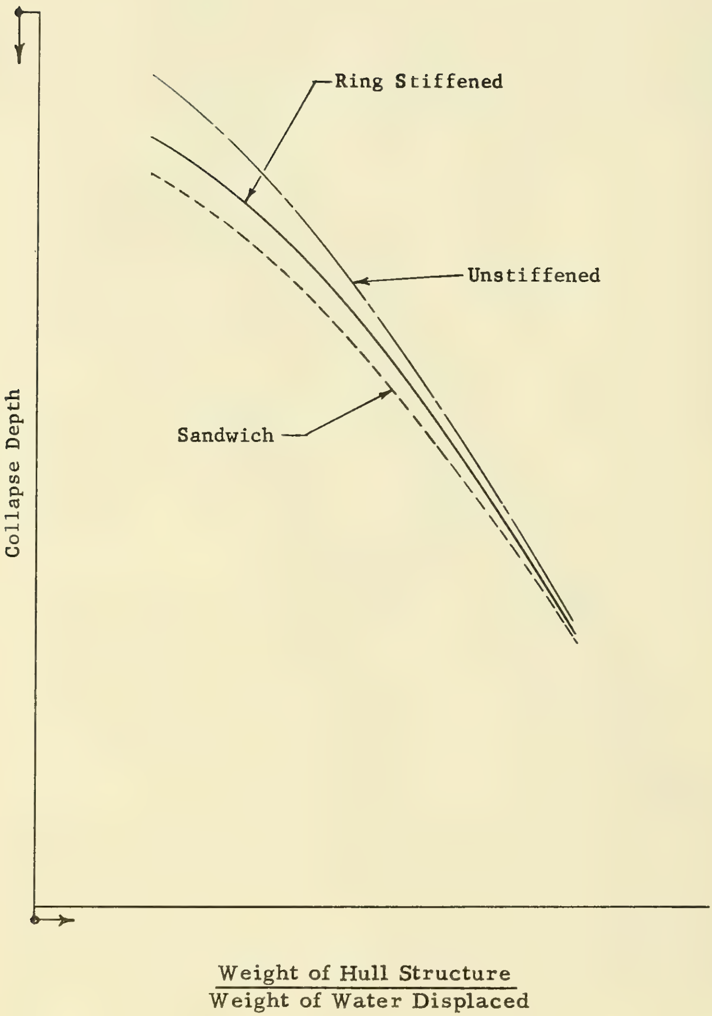


Figure 22 - Strength-Weight Ratios for Different Cylindrical Pressure Hulls

brought out in a qualitative way in Figure 22 where the static strength-weight ratios for unstiffened, ring-stiffened, and sandwich-type cylindrical pressure hulls are compared. If this turns out to be the case, and there is some indication to this effect, then the advantages offered by any of the newer multilayered, sandwich, and sheathed constructions will be mainly ease of fabrication and some possibility of improved dynamic and explosion resistance; these latter considerations have not been discussed in the earlier sections on strength analysis, and will not be pursued here either. Therefore, we can state that the main advantages of these new pressure-hull constructions stem from the use of thinner plating materials with their inherently superior ductility, higher yield strength, greater notch toughness, uniformity of physical and mechanical properties, and greater ease of fabrication.

Krenzke[69] at the Model Basin has conceived the idea of applying the bimetallic, and in general, the sheathed construction idea to cylindrical pressure hulls. A major advantage of this technique is that it permits the use of available materials such as the very high strength titanium and aluminum alloys by eliminating the handicaps of nonweldability and the propensity to corrosion. The basic idea is that forged and/or machined ring segments, not physically joined together, are placed inside a thin, weldable outer jacket made of some compatible material so that upon initial application of pressure, the thin outer jacket is stressed beyond its yield point and thus plastically deforms around the more rigid high-strength

ring segments. Upon release of pressure, the ring segments expand more than the outer jacket so that the two become firmly locked together, the outer jacket being in residual tension and the ring segments comprising the hull structure being in residual compression. This technique offers many possibilities for combining the best properties of high-strength nonweldable materials, and, at the same time, for circumventing the obvious shortcomings of each. Two possible combinations are shown in Figure 21. Short[70] has suggested the use of variable-thickness shell segments as a means of eliminating the longitudinal bending action between adjacent stiffening elements and thus permitting a state of "pure membrane" behavior. This innovation is shown in Figure 21 where a web-stiffened titanium sandwich hull core of varying thickness between webs is encased in a fiberglass jacket.

Another of the more promising structural concepts for the design and fabrication of cylindrical pressure hull structures for deep-depth application is the sandwich concept. Structural engineers in the aircraft industry have long recognized and taken advantage of the favorable strength-weight and thermal-dispersion characteristics of sandwich-type construction in the design of modern high-speed aircraft and missiles. In studying the literature, however, it has been found that the loading conditions encountered in these applications have dictated sandwich structural arrangements that would be of no direct use to the naval architect in the design of pressure hulls for submersibles. Most of this work (e.g., see Reference 71)

has been concerned chiefly with structural arrangements comprised of thin sheets or facings, flat or curved, and spaced apart from each other by a core of low Young's modulus material intended primarily to increase the moment of inertia or bending rigidity of the section. Such cores possess shear-transmitting properties but are rather weak in compression, particularly in the circumferential direction.

The demands of hydrostatic pressure loading are such that in order to exploit the sandwich concept for pressure hull construction, a core comprised of elements which are compression resistant as well as shear resistant is sought. For this latter application, the membrane loads are predominant, whereas in aerospace applications, the bending-type loads are of prime concern. With the thinking directed along these lines, Buhl and Pulos[72] conducted a series of exploratory structural model tests to investigate the strength-weight advantages of sandwich-type cylindrical pressure hulls having "hard" cores. The configurations which have received the most attention include the web-type and tube-type core arrangements. The earlier results found by these investigators indicated that collapse-strength advantages on the order of 25 percent may exist over the conventional ring-stiffened cylinder in certain ranges of geometry; this is shown qualitatively to some extent in Figure 22.

More recent experimental studies[73] conducted at the Model Basin on sandwich cylinders have revealed another possible strength advantage inherent in these type hull structures. Hydrotests of some structural

models made of high-strength steel have collapsed at pressures much higher than those predicted by calculations based on the most optimistic expectations. These observations have been explained on the basis that the exceptionally high buckling strength possessed by these sandwich cylinders permits straining of the material well into the inelastic and work-hardening ranges with its beneficial effects of higher strength levels. Although this phenomenon may exist, it should be viewed as more of a scientific curiosity than one which could be incorporated in a design. Furthermore, this strain-hardening influence may not even exist for sandwich structures fabricated of the higher strength but lower modulus materials like titanium, for example.

In addition to experimental programs on sandwich cylinders, analytical studies have been going on concurrently in order to develop rational formulas based on thin-shell theory for predicting the static structural response of these type structures. In Reference 74 Pulos presents an analysis of the axisymmetric elastic deformations and stresses in a web-stiffened sandwich cylinder under hydrostatic pressure. Raetz presents a similar analysis for the tube-stiffened sandwich cylinder.[75] An examination of available elastic-stability analyses[76,77] for three-layered sandwich cylinders under hydrostatic pressure revealed the absence of satisfactory criteria pertinent to the thick-walled sandwich hull problem in which the core possesses compression- and shear-resistant characteristics. For this reason, a new analysis has been developed

by Pulos and Yang.[78]

The question of how important interlaminar shear resistance is and what influence it has on instability strength of multilayered shell structures should be studied. Tests of structural models having different numbers of layers comprising a cylindrical shell should be conducted to study the relationship of different geometric parameters to ultimate load-carrying capacity. Also, available theories for the stresses in and buckling of multilayered cylinders require examination. An excellent summary of such analyses has been given by Ambartsumian.[79]

A strong possibility exists that underwater vehicles of the future will utilize spherical shells as the main pressure hull or, more likely, to close off the ends of basically cylindrical or spheroidal hulls. Krenzke has recently presented the first results stemming from a major structural research effort on stiffened and unstiffened spherical shells.[80] From tests of machined models, designed to investigate both elastic and inelastic behavior, he has obtained collapse pressures on the order of two to three times the values reported by earlier investigators. The collapse strength of those models which failed elastically could not be predicted by the classical small-deflection theory of Zoelly[81] or any of the available large-deflection theories.[28,82] However, Krenzke was able to achieve collapse pressures which were 70 percent of the predictions from the linear theory of Zoelly. Another significant result reported by Krenzke is that, contrary to the belief of others, the buckling coefficient appears

to be a constant and not a function of the thickness-radius ratio of the sphere; this appears to be true at least for the range of thickness-radius ratios from 0.10 to 0.01 investigated by Krenzke. Tests with welded hemispheres and tests of spherical caps (having included solid-angles ranging from 5 degrees to a full sphere) are also presently underway at the Model Basin.

Although the overall shape of even present-day submersibles is that of a general extended ellipsoid of revolution, (see the ALBACORE (AGSS-569) hull in Figure 1), the main pressure hull structure is still the ring-stiffened, right-circular cylinder. In the future it may be desirable to consider the use of doubly curved shells such as prolate spheroids for the major hull structure. Such shells are superior to the circular cylinder in hydrodynamic characteristics and may also possess significantly greater structural strength for the same weight of hull. At present, both analytical and experimental data on such shell structures are rather limited. Two major problems which require attention are those dealing with the axisymmetric stresses in and buckling of ring-stiffened prolate spheroids. A general set of edge coefficients for a spheroid of constant but different radii of curvature in the two principal directions may serve as a basis for carrying out an analysis of the elastic deformations and stresses in the ring-stiffened structure. Efforts should also be directed toward an examination of available buckling analyses such as those summarized in Reference 83. Structural models are required to

obtain necessary experimental data with which to check the predictions of theory.

Further requirements may dictate the use of other than circular cylindrical pressure hulls, such as those possessing either deep-draft or wide-beam cross sections. To meet these needs and retain the same degree of rationality that exists for the structural analysis and design of circular cylindrical pressure hulls, experimental and analytical studies are required. Under joint sponsorship by the Bureau of Ships and the Office of Naval Research, the group at the Polytechnic Institute of Brooklyn has been conducting theoretical studies of the elastic behavior of noncircular cylindrical shells under hydrostatic pressure. Kempner, Romano, and Vafakos have already reported some of their findings in a series of PIBAL reports.[84-86] One of the more significant results they have found so far is that for the case of a short cylinder having a quasi-elliptic cross section and either simply supported or clamped edges, it appears that one can use the local radius of curvature in the equations for the right-circular cylinder to get excellent prediction of the longitudinal and circumferential stresses in the shell. Although one might argue that intuitively this result is not at all unexpected, it also turns out that no such simple analogy can be deduced for the case of the elastically supported (ring-stiffened) quasi-elliptic cylinder and recourse must be had to the exact solution which is based on a Fourier series approach. As an adjunct to these analytical studies, model tests are required to obtain

experimental data with which to check the predictions of theory.

In conclusion, it is appropriate to cite two references which summarize the valuable theoretical research on the transverse strength of pressure hulls being conducted in the Department of Aerospace Engineering and Applied Mechanics at the Polytechnic Institute of Brooklyn. This research effort has been going on for the past 15 years under the financial sponsorship and technical cognizance of the Office of Naval Research and the Bureau of Ships. Much useful information has resulted from these investigations; some has already been discussed in the main body of this presentation, and a great deal has found its way into present-day methods of structural analysis and design of pressure hulls for submersibles. Prof. Nicholas J. Hoff (presently Head of the Department of Aeronautics and Astronautics at Stanford University) reviewed this work for the period from 1947 through 1952 in PIBAL Report 209. More recently, Kempner[87] has discussed in considerable detail the research effort covering the period from 1952 through 1961. The topics are varied and include circular cylindrical shells reinforced by nonuniform frames; noncircular cylindrical shells; inelastic behavior of circular cylindrical shells, disks, and frames; large deflections of circular cylindrical shells; and general instability of reinforced circular cylindrical shells.

Finally, a valuable bibliography on shells and shell-like structures has been compiled by Nash.[88] This document has proved to be a very useful

source of information to structural analysts and designers of pressure vessels in general.

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1. The author of the enclosed report was invited as a guest lecturer at a short course entitled, "Analysis and Design of Modern Pressure Vessels," held at the University of California in Los Angeles during 16-27 July, 1962. The series of lectures presented by the author has been compiled in the form of a David Taylor Model Basin Report, and is being forwarded as enclosure (1).

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This report reviews the "state of the art" in the field of pressure hull structural analysis. Equations and formulas developed from considerations of thin-shell theory to describe the elastic and inelastic behavior of ring-stiffened cylindrical shells under the action of hydrostatic pressure are summarized.

No direct comparison between theory and experiment is included herein; however, only those analyses are included which represent the most up-to-date knowledge, in the opinion of David Taylor Model Basin structural analysts, and which have found firm experimental confirmation. This presentation should not be interpreted as representing an exhaustive review of all

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